By condition ii) of Theorem 2, the switched system Σ_σ is exponentially γ_σ-stable if $\tau_D \geq \ln \bar{m}/(\underline{\gamma} - \gamma_\sigma)$, choosing $\gamma_\sigma = 0.001$, one has $\tau_D \geq 153$.

Example 2: A single-mode plant is here considered assuming the same dynamical matrix $A(\theta(t))$ considered in [14]

$$A(\theta(t)) = \begin{bmatrix} -1 - 1.3\theta(t) & -0.5 - 20\theta(t) \\ -1 + 2\theta(t) & -2 - 10\theta(t) \end{bmatrix}.$$

In the above reference, it is shown that $A(\theta(t))$ is quadratically stable $\forall\theta(t)$ such that $\theta(t) \in [0, 1]$, $\forall t \geq 0$. Assuming $\theta(t) = [a, b] + [c, d]t + [e, f]t^2$, the corresponding $A(\theta(t))$ can be written in the form (2) with $\ell = 2$, $n_0 = n_1 = n_2 = 2$. If $[a, b] = [-2, 1]$, $[c, d] = [-5, 6]$, $[d, e] = [1, 2]$, it is found that the whole set of LMIs (13)–(21) is not satisfied so that (5) does not hold $\forall t \geq 0$. Nevertheless, it is found that inequalities (12) are satisfied starting from $\bar{k} = 3$. The relative LMIs are given by the strict version of (21) with $i = 1$, $j = 2 = \ell$ (corresponding to $H_3(\alpha) < 0$), and by (16), (17) with $i = 2 = \ell$ (corresponding to $H_4(\alpha) \leq 0$). By Lemma 2, $\dot{V}(x(t), \alpha) < 0$, $\forall t \geq \bar{t} \triangleq \bar{t}(\bar{k}) = \bar{t}(3)$, and the value $\bar{t}(3) = 12$ has been numerically found. It follows that, under the assumed polynomial behavior of $\theta(t)$, the exponential stability of $A(\theta(t))$ is attained also for arbitrarily large variations of the parameter and of its variation rate.

V. Conclusion

The stability analysis developed in this technical note is based on the mild modeling assumption of a dynamical matrix with smoothly time varying elements. This made it possible to consider the significant class of switching linear systems with uncertain elements whose time behavior is described by piecewise interval polynomial of arbitrary degree. This in turn implies that the present approach offers the possibility of dealing with uncertain plants whose parameters are not confined inside a relatively small polytopic region. Unlike all the other methods, both parameters and their derivatives may take values over arbitrarily large sets and theoretically unbounded dynamical matrices can be considered. The sufficient LMIs stability conditions have been derived using a quadratic Lyapunov function polinomially depending on time. It has also been shown that, giving up the requirement $\dot{V}_p(x(t), \alpha^{(p)}) < 0, \forall t \in T_k^{(p)} \in \{T_k^{(p)}\}$, reduced conservatism conditions are obtained because the number of LMIs reduces to $n_\ell + n_\ell(n_\ell - 1)/2$, independently of the polynomial degree ℓ.

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Non-Linear Symmetry-Preserving Observers on Lie Groups

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***Abstract*—In this technical note, we give a geometrical framework for the design of observers on finite-dimensional Lie groups for systems which possess some specific symmetries. The design and the error (between true and estimated state) equation are explicit and intrinsic. We consider also a particular case: left-invariant systems on Lie groups with right equivariant output. The theory yields a class of observers such that the error equation is autonomous. The observers converge locally around *any* trajectory, and the global behavior is independent from the trajectory, which is reminiscent of the linear stationary case.**

***Index Terms*—Inertial navigation, invariance, Lie group, nonlinear asymptotic observer, symmetry.**

I. Introduction

Symmetries (invariances) have been used to design controllers and for optimal control theory ([7], [8], [11], [16], [18]), but far less for the design of observers. References [2] and [4] develop a theory of symmetry-preserving observers and presents three nonlinear observers for three examples of engineering interest: a chemical reactor, a nonholonomic car, and an inertial navigation system. In the two latter examples the state space and the group of symmetry have the same dimension and (since the action is free) the state space can be identified with the group (up to some discrete group). Once again, there is extensive literature on control systems on Lie groups (see [10] as one of the pioneering papers), but far less on observers on Lie groups ([5], [9], [12]). Applying the general theory to the Lie group case, we develop here a proper theory of symmetry-preserving observers on Lie groups. The

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advantage over [2] and [4] is that the observer design is explicit (the implicit function theorem is not needed) and intrinsic, the error equation and its first-order approximation can be computed explicitly, and are intrinsic, and all the formulas are globally defined. Moreover, this technical note is a step further in the symmetry-preserving observers theory since [2], [4] does not deal at all with convergence issues in the general case. Here, using the explicit error equation, we introduce a new class of trajectories around which we build convergent observers. In the case of Section III, a class of first-order convergent observers around any trajectory is given. The theory applies to various systems of engineering interest modeled as invariant systems on Lie groups, such as cart-like vehicles and rigid bodies in space. In particular, it is well suited to attitude estimation and some inertial navigation examples.

The technical note is organized as follows: in Section II we give a general framework for symmetry-preserving observers on Lie groups. It explains the general form of the observers [12], [9], [6], and [2], [4] (car example) based on the group structure of SO(3) and (respectively) SE(2), without considering the convergence issues. The design, the error equation, and its first-order approximation are given explicitly. It is theoretically explained why the error equation in the car example of [2] and [4] does not depend on the trajectory (although it depends on the inputs). Then we introduce a new class of trajectories called permanent trajectories which extend the notion of equilibrium point for systems with symmetries: making a symmetry-preserving observer around such a trajectory boils down to making a Luenberger observer around an equilibrium point. We characterize permanent trajectories geometrically and give a locally convergent observer around any permanent trajectory.

In Section III, we consider the special case of a left-invariant system with right equivariant output. It can be looked at as the motion of a generalized rigid body in space with measurements expressed in the body-fixed frame, as it will be explained in Section III-A-1. Thus, it applies to some inertial navigation examples. In particular, it allows to explain theoretically why the error equation in the inertial navigation example of [2], [4] is autonomous. A class of first-order convergent observers such that the error equation is autonomous is derived. This property is reminiscent of the linear stationary case. We also explore the links between right equivariance of the output map and observability.

Preliminary versions of Section III can be found in [3] and [5].

II. Symmetry-Preserving Observers on Lie Groups

A. Invariant Observer and Error Equation

Consider the following system

$$\frac{d}{dt}x(t) = f(x,u) \tag{1}$$

$$y = h(x,u) \tag{2}$$

where x is an element of a Lie group G of dimension n, $u \in \mathcal{U} = \mathbb{R}^m$, $y \in \mathcal{Y} = \mathbb{R}^p$ (the whole theory can be easily adapted to the case where $\mathcal{U}$ and $\mathcal{Y}$ are smooth m and p-dimensional manifolds, for instance Lie groups), and f is a smooth vector field on G. $u \in \mathcal{U}$ is a known input (control, measured perturbation, constant parameter, time t i.e. $f(x,u) = f(x,t)$ etc.).

Definition 1: Let G be a Lie Group with identity e and Σ an open set (or more generally a manifold). A *left* group action $(\phi_g)_{g\in G}$ on Σ is a smooth map

$$(g,\xi) \in G \times \Sigma \mapsto \phi_g(\xi) \in \Sigma$$

such that:

- $\phi_e(\xi) = \xi$ for all ξ;
- $\phi_{g_2}(\phi_{g_1}(\xi)) = \phi_{g_2 g_1}(\xi)$ for all g_1, g_2, ξ.

In analogy, one defines a right group action the same way except that $\phi_{g_2}(\phi_{g_1}(\xi)) = \phi_{g_1 g_2}(\xi)$ for all g_1, g_2, ξ. Suppose G acts on the left on $\mathcal{U}$ and $\mathcal{Y}$ via $\psi_g : \mathcal{U} \to \mathcal{U}$ and $\rho_g : \mathcal{Y} \to \mathcal{Y}$. Suppose the dynamics (1) is invariant in the sense of [2], [4] where the group action on the state space (the group itself) is made of left multiplication: for any $g \in G$, $DL_g f(x,u) = f(gx, \psi_g(u))$, i.e.,

$$\forall\, x, g \in G \;\; f\left(L_g(x), \psi_g(u)\right) = DL_g f(x,u)$$

where $L_g : x \mapsto gx$ is the left multiplication on G, and DL_g the induced map on the tangent space. DL_g maps the tangent space $TG|_x$ to $TG|_{gx}$. Let $R_g : x \mapsto xg$ denote the right multiplication and DR_g its induced map on the tangent space. As in [2] and [4], we suppose that the output $y = h(x,u)$ is equivariant, i.e., $h(\varphi_g(x), \psi_g(u)) = \rho_g(h(x,u))$ for all g, x, u.

Definition 2: Consider the change of variables $X = gx, U = \psi_g(u)$ and $Y = \rho_g(y)$. The system (1), (2) is left-invariant with equivariant output if for all $g \in G$ it is unaffected by the latter transformation: $(d/dt)X(t) = f(X,U), Y = h(X,U)$.

We are going to build observers which respect the symmetries (left-invariance under the group action) adapting the constructive method of [2] and [4] to the Lie group case.

1) Invariant Pre-Observers: Following [17] (or [2] and [4]) consider the action $(\phi_g)_{g\in G}$ of G on $\Sigma = \mathbb{R}^s$ where s is any positive integer. Let $(x,z) \in G \times \mathbb{R}^s$, one can compute (at most) s functionally independent scalar invariants of the variables (x,z) the following way: $I(x,z) = \phi_{x^{-1}}(z) \in \mathbb{R}^s$. It has the property that any invariant real-valued function $J(x,z)$ which verifies $J(gx, \phi_g(z)) = J(x,z)$ for all g, x, z is a function of the components of $I(x,z) : J(x,z) = \mathcal{H}(I(x,z))$. Applying this general method, we find a complete set of invariants of $(x,u) \in G \times \mathcal{U}$

$$I(x,u) = \psi_{x^{-1}}(u) \in \mathcal{U}. \tag{3}$$

Take n linearly independent vectors $(W_1, \ldots, W_n)$ in $TG|_e = \mathfrak{g}$, the Lie algebra of the group G. Define n vector fields by the invariance relation $w_i(x) = DL_x W_i \in TG|_x, i = 1\ldots n, x \in G$. The vector fields form an invariant frame [17] according to [2] and [4].

Definition 3 (Pre-Observer): The system $(d/dt)\hat{x} = F(\hat{x},u,y)$ is a *pre-observer* of (1) and (2) if $F(x,u,h(x,u)) = f(x,u)$ for all $(x,u) \in G \times \mathcal{U}$.

The definition does not deal with convergence; if moreover $x(t)^{-1}\hat{x}(t) \to e$ as $t \to +\infty$ for every (close) initial conditions, the pre-observer is an (asymptotic) *observer*. It is called *G-invariant* if $F(g\hat{x}, \psi_g(u), \varrho_g(y)) = DL_g F(\hat{x},u,y)$ for all $(g,\hat{x},u,y) \in G \times G \times \mathcal{U} \times \mathcal{Y}$.

Lemma 1: Any invariant pre-observer reads

$$\frac{d}{dt}\hat{x} = f(\hat{x},u) + DL_{\hat{x}}\left(\sum_{i=1}^{n} \mathcal{L}_i\left(\psi_{\hat{x}^{-1}}(u), \rho_{\hat{x}^{-1}}(y)\right) W_i\right) \tag{4}$$

where the $\mathcal{L}_i$ are any smooth functions of their arguments such that $\mathcal{L}_i(\psi_{\hat{x}^{-1}}(u), h(e, \psi_{\hat{x}^{-1}}(u))) = 0$. The proof of this lemma is analogous to [2] and [4]: one can write $DL_{\hat{x}^{-1}}((d/dt)\hat{x} - f(\hat{x},u)) = \sum_{i=1}^{n} \mathcal{F}_i(\hat{x},u,y) W_i \in \mathfrak{g}$, where the $\mathcal{F}_i's$ are invariant scalar functions of their arguments. But a complete set of invariants of $\hat{x}, u, y$ is made of the components of $(\psi_{\hat{x}^{-1}}(u), \rho_{\hat{x}^{-1}}(y))$, thus $\mathcal{F}_i(\hat{x},u,y) = \mathcal{L}_i(\psi_{\hat{x}^{-1}}(u), \rho_{\hat{x}^{-1}}(y))$. And when $\hat{x} = x$, we have $\rho_{\hat{x}^{-1}}(y) = h(\hat{x}^{-1}x, \psi_{\hat{x}^{-1}}(u)) = h(e, \psi_{\hat{x}^{-1}}(u))$ and the $\mathcal{L}_i$'s cancel.

2) Invariant State-Error Dynamics: Consider the invariant state-error $\eta = x^{-1}\hat{x} \in G$. It corresponds to the error between $\hat{x}$ and x in the sense of the group multiplication (the usual linear error $\hat{x} - x$ is not defined on G). It is invariant by left multiplication: $\eta = (gx)^{-1}(g\hat{x})$ for any $g \in G$. Notice that a small error corresponds to η close to e.

Contrarily to [2], [4], the time derivative of η can be computed explicitly. We recall R_g denotes the right multiplication map on G. Since we have

- for any $g_1, g_2 \in G$, $DL_{g_1}DL_{g_2} = DL_{g_1 g_2}$, $DR_{g_1}DR_{g_2} = DR_{g_2 g_1}$, $DL_{g_1}DR_{g_2} = DR_{g_2}DL_{g_1}$;
- $I(\hat{x}, u) = \psi_{\hat{x}^{-1}}(u) = \psi_{(x\eta)^{-1}}(u)$;
- $\rho_{\hat{x}^{-1}}(h(x,u)) = h(\hat{x}^{-1}x, \psi_{\hat{x}^{-1}}(u))$ writes $\rho_{\hat{x}^{-1}}(y) = h(\eta^{-1}, \psi_{(x\eta)^{-1}}(u))$;
- $(d/dt)\eta = (d/dt)(x^{-1}\hat{x}) = DL_{x^{-1}}(d/dt)\hat{x} - DR_{\hat{x}}(d/dt)x^{-1}$ with $(d/dt)x^{-1} = -DL_{x^{-1}}DR_{x^{-1}}(d/dt)x$;

the error dynamics reads

$$\frac{d}{dt}\eta = DL_\eta f\left(e, \psi_{(x\eta)^{-1}}(u)\right) - DR_\eta f\left(e, \psi_{x^{-1}}(u)\right) + DL_\eta\left(\sum_{i=1}^{n}\mathcal{L}_i\left(\psi_{(x\eta)^{-1}}(u), h\left(\eta^{-1}, \psi_{(x\eta)^{-1}}(u)\right)\right)W_i\right). \quad (5)$$

The invariant error η obeys a differential equation that is coupled to the system trajectory $t \mapsto (x(t), u(t))$ only via the invariant term $I(x,u) = \psi_{x^{-1}}(u)$. Note that when $\psi_g(u) \equiv u$ the invariant error dynamics is independent of the state trajectory $x(t)$! This is why we have this property in the nonholonomic car example of [2], [4].

3) Invariant First-Order Approximation: For η close to e, one can set in (5) $\eta = \exp(\epsilon\xi)$ where ξ is an element of the Lie algebra $\mathfrak{g}$ and $\epsilon \in \mathbb{R}$ is small. The linearized invariant state error equation can always be written in the same tangent space $\mathfrak{g}$. We recall that for $\zeta, \varsigma \in \mathfrak{g}$, the Lie bracket of $\mathfrak{g}$ (denoted by "[,]") is defined by $[\zeta, \varsigma] = \mathrm{ad}_\zeta \varsigma$ where

$$\mathrm{ad}_\zeta : \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{ad}_\zeta = \frac{d}{ds}|_{s=0} DR_{\exp(s\zeta)^{-1}} DL_{\exp(s\zeta)}$$

Also using the fact that $(DL_\eta - Id)$ is a term of order ϵ, that $\psi_{(x\eta)^{-1}}(u)$ can be approximated by $\psi_{x^{-1}}(u)$ up to terms of order ϵ, and that $\eta^{-1} = \exp(-\epsilon\xi)$, we have up to second-order terms in ϵ

$$\begin{aligned}\frac{d}{dt}\xi = {} & [\xi, f(e, \psi_{x^{-1}}(u))] - \frac{\partial f}{\partial u}(e, \psi_{x^{-1}}(u)) \\ & \times \frac{\partial \psi}{\partial g}(e, \psi_{x^{-1}}(u))\,\xi \\ & - \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}_i}{\partial h}(\psi_{x^{-1}}(u), \right. \\ & \left. h(e, \psi_{x^{-1}}(u))\,\frac{\partial h}{\partial x}(e, \psi_{x^{-1}}(u))\,\xi\right) W_i \end{aligned} \quad (6)$$

where ψ is viewed as a function of (g, u), and $\partial\mathcal{L}_i/\partial h$ denotes the partial derivative of $\mathcal{L}_i$ with respect to its second argument. The gains $(\partial\mathcal{L}_i/\partial h)(\psi_{x^{-1}}(u), h(e, \psi_{x^{-1}}(u))$ can be tuned via linear techniques to achieve local convergence.

B. Local Convergence Around Permanent Trajectories

The aim of this paragraph is to extend local convergence results around an equilibrium point to a class of trajectories we call permanent trajectories.

Definition 4: A trajectory of (1) is permanent if $I(x(t), u(t)) = \bar{I}$ is independent of t.

Note that adapting this definition to the general case of symmetry-preserving observers [2], [4] is straightforward. Any trajectory of the system verifies $(d/dt)x(t) = DL_{x(t)}f(e, \psi_{x(t)^{-1}}(u(t)))$ thanks to the invariance of the dynamics. It is permanent if $I(x(t), u(t)) = \psi_{x^{-1}(t)}(u(t)) = \bar{u}$ is independent of t. The permanent trajectory $x(t)$ is then given by $x(0)\exp(t\bar{w})$ where $\bar{w}$ is the left invariant vector field associated to $f(e, \bar{u})$. Thus $x(t)$ corresponds, up to a left translation defined by the initial condition, to a one-parameter subgroup.

Let us make an observer around an arbitrary permanent trajectory: denote by $(x_r(t), u_r(t))$ a permanent trajectory associated to $\bar{u} = \psi_{\bar{x}_r^{-1}(t)}u_r(t)$. Let us suppose we made an invariant observer following (4). Then the error (5) writes

$$\frac{d}{dt}\eta = DL_\eta f\left(e, \psi_{\eta^{-1}}(\bar{u})\right) - DR_\eta f(e, \bar{u}) + DL_\eta\left(\sum_{i=1}^{n}\mathcal{L}_i\left(\psi_{\eta^{-1}}(\bar{u}), h\left(\eta^{-1}, \psi_{\eta^{-1}}(\bar{u})\right)\right)W_i\right) \quad (7)$$

since $\psi_{(x_r\eta)^{-1}}(u) = \psi_{\eta^{-1}}(\psi_{x_r^{-1}}(u)) = \psi_{\eta^{-1}}(\bar{u})$. The first-order approximation (6) is now a time-invariant system

$$\frac{d}{dt}\xi = [\xi, f(e, \bar{u})] - \frac{\partial f}{\partial u}(e, \bar{u})\frac{\partial \psi}{\partial g}(e, \bar{u})\xi - \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}_i}{\partial h}(\bar{u}, h(e, \bar{u}))\frac{\partial h}{\partial x}(e, \bar{u})\xi\right)W_i.$$

Let us write ξ and $f(e,u)$ in the frame defined by the W_i's: $\xi = \sum_{k=1}^{n}\xi^k W_k$ and $f(e, \bar{u}) = \sum_{k=1}^{n}\bar{f}^k W_k$. Denote by C_{ij}^k the structure constants associated with the Lie algebra of G : $[W_i, W_j] = \sum_{k=1}^{n} C_{ij}^k W_k$. The above system reads

$$\frac{d}{dt}\xi = (A + \bar{\mathcal{L}}C)\xi \quad (8)$$

where

$$\begin{aligned} A &= \left(\sum_{k=1}^{n} C_{jk}^i \bar{f}^k - \left[\frac{\partial f}{\partial u}(e, \bar{u})\frac{\partial \psi}{\partial g}(e, \bar{u})\right]_{i,j}\right)_{1\le i,j\le n}, \\ \bar{\mathcal{L}} &= \left(-\frac{\partial \mathcal{L}_i}{\partial h_k}(\bar{u}, h(e, \bar{u}))\right)_{\substack{1\le i\le n\\ 1\le k\le p}}, \\ C &= \left(\frac{\partial h_k}{\partial x_j}(e, \bar{u})\right)_{\substack{1\le k\le p\\ 1\le j\le n}} \end{aligned}$$

and where $(x_1, \ldots, x_n)$ are the local coordinates around e defined by the exponential map: $x = \exp(\sum_{i=1}^{n} x_i W_i)$. If we assume that the pair (A, C) is observable we can choose the poles of $A + \bar{\mathcal{L}}C$ to get an invariant and locally convergent observer around any permanent trajectory associated to $\bar{u}$. Let $W(x) = [W_1(x), \ldots, W_n(x)]$. It suffices to take

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u(t)) + W(\hat{x})\bar{\mathcal{L}}\rho_{\hat{x}^{-1}}(y(t)). \quad (9)$$

Examples: In the nonholonomic car example of [2] and [4], permanent trajectories are made of lines and circles with constant speed. In the inertial navigation example of [2], [4], $\psi_{x^{-1}}(u) = \left(\begin{smallmatrix} q*\omega*q^{-1} \\ q*(a+v\times\omega)*q^{-1}\end{smallmatrix}\right)$, a trajectory is permanent if $q*\omega*q^{-1}$ and $q*(a+v\times\omega)*q^{-1}$ are independent of t. Some computations show that any permanent trajectory reads

$$\begin{aligned} q(t) &= \exp\left(\frac{\Omega}{2}t\right) * q_0 \\ v(t) &= q_0^{-1} * \left((\lambda\Omega t + \Upsilon + \exp\left(-\frac{\Omega}{2}t\right) * \Gamma * \exp\left(\frac{\Omega}{2}t\right)\right) * q_0 \end{aligned}$$

where Ω, Υ, and Γ are constant vectors of $\mathbb{R}^3$, λ is a constant scalar, and q_0 is a unit-norm quaternion. These constants can be arbitrarily chosen. Hence, the general permanent trajectory corresponds, up to a Galilean transformation, to a helicoidal motion uniformly accelerated along the rotation axis when $\lambda \neq 0$; when λ tends to infinity and Ω to 0, we recover as a degenerate case a uniformly accelerated line. When $\lambda = 0$ and $\Gamma = 0$ we recover a coordinated turn.

III. Left Invariant Dynamics and Right Equivariant Output

A. Invariant Observer and Error Equation

1) Left Invariant Dynamics and Right Equivariant Output: Consider the following system:

$$\frac{d}{dt}x(t) = f(x,t) \tag{10}$$

$$y = h(x) \tag{11}$$

where we still have $x \in G$, $y \in \mathcal{Y}$, and f is a smooth vector field on G. Let us suppose the dynamics (10) is *left*-invariant (see, e.g., [1]), i.e., : $\forall g, x \in G f(L_g(x), t) = DL_g f(x,t)$. For all $g \in G$, the transformation $X(t) = gx(t)$ leaves the dynamics equations unchanged: $(d/dt)X(t) = f(X(t), t)$. Let $\omega_s = DL_{x^{-1}}(d/dt)x \in \mathfrak{g}$. Then the dynamics (10) are given by (12), generalizing the motion equation of a rigid body in space fixed at a point (16). This is why it is stated in [1] that one can look at any left invariant dynamics on G as a motion of a "generalized rigid body" with configuration space G. Thus one can look at $\omega_s(t) = f(e,t)$ as the "angular velocity in the body", where e is the group identity element (whereas $DR_{x^{-1}}(d/dt)x$ is the "angular velocity in space"). We will systematically write the left-invariant dynamics (10)

$$\frac{d}{dt}x(t) = DL_x\omega_s(t). \tag{12}$$

Let us suppose that $h : G \to Y$ is a *right* equivariant smooth output map. The group action on itself by right multiplication corresponds to the transformations $(\rho_g)_{g\in G}$ on the output space $\mathcal{Y}$: for all $x, g \in G$, $h(xg) = \rho_g(h(x))$ i.e.,

$$h\left(R_g(x)\right) = \rho_g\left(h(x)\right).$$

Left multiplication corresponds then for the generalized body to a change of space-fixed frame, and right multiplication to a change of body-fixed frame. If all the measurements correspond to some part of the state x expressed in the body-fixed frame, they are affected by a change of body-fixed frame, and the output map is right equivariant. Thus the theory allows to build nonlinear observers such that the error equation is *autonomous*, in particular for *cart-like vehicles* and *rigid bodies in space* (according to the Eulerian motion) with *measurements in the body-fixed frame* (see the example below).

2) Observability: If the dimension of the output space is strictly smaller than the dimension of the state space $(\dim y < \dim g)$ the system is necessarily not observable. This comes from the fact that, in this case, there exists two distinct elements x_1 and x_2 of G such that $h(x_1) = h(x_2)$. If $x(t)$ is a trajectory of the system, we have $(d/dt)x(t) = DL_g\omega_s(t)$ and because of the left-invariance, $g_1x(t)$ and $g_2x(t)$ are also trajectories of the system

$$\frac{d}{dt}\left(g_1x(t)\right) = DL_{g_1x}\omega_s(t), \quad \frac{d}{dt}\left(g_2x(t)\right) = DL_{g_2x}\omega_s(t).$$

But since h is right equivariant: $h(g_1x(t)) = \rho_{x(t)}h(g_1) = \rho_{x(t)}h(g_2) = h(g_2x(t))$. The trajectories $g_1x(t)$ and $g_2x(t)$ are distinct and for all t they correspond to the same output. The system is unobservable.

3) Applying the General Theory of Section II: There are two ways to apply the theory of Section II. i) The most natural (respecting left-invariance) does not yield the most interesting properties: let $\mathcal{U} = \mathbb{R} \times \mathcal{Y}$ and let us look at $(u_1, u_2) = (t, h(e))$ as inputs. For all $g \in G$, let $\psi_g(t, h(e)) = (t, \rho_{g^{-1}}(h(e)))$. Define a new output map $H(x,u) = h(x) = \rho_x(h(e)) = \rho_x(u_2)$. It is unchanged by the transformation introduced in definition 2 since $H(X,U) = \rho_{gx}(\rho_{g^{-1}}(u_2)) = H(x,u)$ for all $g \in G$. (10), (11) is then a left-invariant system in the sense of definition 2, when the output map is $H(x,u)$. ii) Let us rather look at $\omega_s(t)$ as an input: $u(t) = \omega_s(t) \in \mathcal{U}$, where $\mathcal{U} = \mathfrak{g} \equiv \mathbb{R}^n$ is the input space. Let us define for all g the map $\psi_g : G \to \mathcal{U}$ the following way:

$$\psi_g = DL_{g^{-1}}DR_g.$$

It means ψ_g is the differential of the interior automorphism of G. The dynamics (10) writes $(d/dt)x = F(x,u) = DL_xu$ and can be viewed as a right-invariant dynamics. For all x, g we have indeed: $(d/dt)R_g(x) = DR_gDL_x\omega_s(t) = DL_xDL_gDL_{g^{-1}}DR_g\omega_s(t) = DL_{R_g(x)}\psi_g(\omega_s(t)) = F(R_g(x), \psi_g(u))$. Note also that $(\psi_g)_{g\in G}$ and $(\rho_g)_{g\in G}$ are *right* group actions since for all $g_1, g_2 \in G$, we have $\psi_{g_1} \circ \psi_{g_2} = \psi_{g_2g_1}$ and $\rho_{g_1} \circ \rho_{g_2} = \rho_{g_2g_1}$. Thus, we strictly apply the general theory of II, exchanging the roles of left and right multiplication.

4) Construction of the Observers: Take n linearly independent vectors $(W_1, \ldots, W_n)$ in $TG|_e = \mathfrak{g}$. Consider the class of observers of the form

$$\frac{d}{dt}\hat{x} = DL_{\hat{x}}\omega_s(t) + DR_{\hat{x}}\left(\sum_{i=1}^{n}\mathcal{L}_i\left(\rho_{\hat{x}^{-1}}(y)\right)W_i\right) \tag{13}$$

where the $\mathcal{L}_i$'s are smooth scalar functions such that $\mathcal{L}_i(h(e)) = 0$. They are invariant under the transformations defined in Section III-A.3ii).

5) State-Error Dynamics: The error (invariant by right multiplication) is $G \ni \eta = (\hat{x}x^{-1}) = L_{\hat{x}}(x^{-1})$. The error equation is an *autonomous* differential equation (14) independent from the trajectory $t \mapsto x(t)$ (as in the linear stationary case)

$$\frac{d}{dt}\eta = DR_\eta\left(\sum_{i=1}^{n}\mathcal{L}_i\left(h(\eta^{-1})\right)W_i\right). \tag{14}$$

It can be deduced from (5) or directly computed using $(d/dt)\eta = DL_{\hat{x}}((d/dt)x^{-1}) + D_xL_{\hat{x}}(x^{-1})(d/dt)\hat{x}$ and the following points:

- $D_xL_{\hat{x}}(x^{-1})(d/dt)\hat{x} = DR_{x^{-1}}((d/dt)\hat{x}) = DR_{x^{-1}}DL_{\hat{x}}\omega_s(t) + DR_{x^{-1}}DR_{\hat{x}}\sum_{i=1}^{n}\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y))W_i$ which can be written $DR_{x^{-1}}DL_{\hat{x}}\omega_s(t) + DR_\eta\sum_{i=1}^{n}\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y))W_i$;
- moreover, we have $\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y)) = \mathcal{L}_i(\rho_{\hat{x}^{-1}}(h(x))) = \mathcal{L}_i(h(\eta^{-1}))$;
- finally, $DL_{\hat{x}}((d/dt)x^{-1}) = -DL_{\hat{x}}DR_{x^{-1}}DL_{x^{-1}}\dot{x} = -DL_{\hat{x}}DR_{x^{-1}}\omega_s = -DR_{x^{-1}}DL_{\hat{x}}\omega_s(t)$.

6) First Order Approximation: We suppose that η is close to e. Let $\xi \in \mathfrak{g}$ such that $\eta = \exp(\epsilon\xi)$ with $\epsilon \in \mathbb{R}$ small. We have up to second-order terms in ϵ

$$\frac{d}{dt}\xi = -\sum_{i=1}^{n}\left(\frac{\partial\mathcal{L}_i}{\partial h}\left(h(e)\right)\frac{\partial h}{\partial x}(e)\xi\right)W_i.$$

Let us define a scalar product on the tangent space $\mathfrak{g}$ at e, and let us consider the adjoint operator of $Dh(e)$ in the sense of the metrics associated to the scalar product. The adjoint operator is denoted by $(Dh(e))^T$ and we take $\mathcal{L}(y) = K(Dh(e))^T(y - h(e))$. The first-order approximation writes

$$\dot{\xi} = -KDh^TDh\xi \tag{15}$$

and for $K > 0$, admits as a Lyapunov function $\|\xi\|^2$, which is the length of ξ in the sense of the scalar product.

B. A Class of Nonlinear First-Order Convergent Observers

Consider for (10) and (11) the following observers: $(d/dt)\hat{x} = DL_{\hat{x}}\omega_s(t) + DR_{\hat{x}}[\sum_{i=1}^{n}[\mathcal{L}_i(\rho_{\hat{x}}^{-1}(h(x)))]W_i]$ where the $\mathcal{L}_i$'s are smooth scalar functions such that $\mathcal{L}_i(h(e)) = 0$. Using the first-order approximation design, take $\mathcal{L}_1, \ldots, \mathcal{L}_n$ such that the symmetric part (in the sense of the scalar product chosen on $TG|_e$) of the linear map $\xi \mapsto -\sum_{i=1}^{n}((\partial\mathcal{L}_i/\partial h)(h(e))(\partial h/\partial x)(e)\xi)W_i$ is negative. When it is negative definite, we get locally exponentially convergent nonlinear observers around any system trajectory.

IV. Brief Example: Magnetic-Aided Attitude Estimation

To illustrate the theory we present one of the simplest example: magnetic-aided inertial navigation as considered in [5], [13]. To pilot a flying body requires at least a good knowledge of its orientation. This holds for manual, or semi automatic or automatic piloting. In low-cost or "strap-down" navigation systems the measurements of angular velocity $\vec{\omega}$ and acceleration $\vec{a}$ by rather cheap gyrometers and accelerometers are completed by a measurement of the earth magnetic field $\vec{B}$. These various measurements are fused (data fusion) according to the motion equations of the system. The estimation of the orientation is generally performed by an extended Kalman filter. But the following proposed observer, in addition to its structural properties, is simpler. Indeed a more realistic situation is studied along the same lines in [14], [15], with more emphasis on tuning and experimental results; [15] also presents an implementation on an 8-bit microcontroller running at 11 MHz and using the standard C floating point emulation, which illustrates the computational simplicity of the proposed observer.

The orientation (attitude) can be described by an element of the group of rotations SO(3), which is the configuration space of a body fixed at a point. The motion equation is

$$\frac{d}{dt}R = R(\vec{\omega} \times \cdot) \tag{16}$$

where $R \in \mathrm{SO}(3)$ is the matrix which represents the rotation mapping the body-fixed frame to the earth-fixed frame, $\vec{\omega}(t)$ is the instantaneous angular velocity vector measured by the gyroscopes and $(\vec{\omega} \times \cdot)$ is the skew-symmetric matrix corresponding to wedge product with $\vec{\omega}$. If the output is the earth magnetic field $\vec{B}$ measured by the magnetometers *in the body-fixed frame* $y = R^{-1}\vec{B}$ ([6]), it is right equivariant. The output has then dimension 2 (the norm of y is constant) and the state space has dimension 3. Thus the system is not observable according to Section III-A.2. This is why we make an additional assumption as in [5], [13]. Indeed the accelerometers measure $\vec{a} = (d/dt)\vec{v} + R^{-1}\vec{G}$ where $(d/dt)\vec{v}$ is the acceleration of the center of mass of the body and $\vec{G}$ is the gravity vector. We suppose the acceleration of the center of mass is small with respect to $\|\vec{G}\|$ (quasi-stationary flight). The measured output is thus $y = (y_G, y_B) = (R^{-1}\vec{G}, R^{-1}\vec{B})$. One can apply the theory as described in Section III-A.3i) or Section III-A.3ii). This latter section yields a class of first-order convergent observers around any trajectory

$$\begin{aligned}\frac{d}{dt}\hat{R} = \hat{R}(\vec{\omega} \times \cdot) - K_1 \left[\left(\hat{R}y_B \times (\hat{R}y_B - B)\right) \times \cdot\right] \hat{R} \\ -K_2 \left[\left(\hat{R}y_G \times (\hat{R}y_G - G)\right) \times \cdot\right] \hat{R}.\end{aligned}$$

Indeed, it corresponds to (13) with a choice of the $\mathcal{L}_i$'s motivated by the following first-order analysis. If $\eta = \hat{R}R^{-1}$ is the error we have $(d/dt)\eta = -K_1[(\eta B \times (\eta B - B)) \times \cdot]\eta - K_2[(\eta G \times (\eta G - G)) \times \cdot]\eta$. Let us linearize this equation. As the tangent space to SO(3) at the identity rotation Id is made of skew-symmetric matrices we write for η close to $Id : \eta = Id + \xi \times \cdot$ with $\xi \in \mathbb{R}^3$ small. Up to second-order terms in ξ we have $(d/dt)\xi = -K_1 B \times (\xi \times B) - K_2 G \times (\xi \times G)$. Without loss of generality, one can assume that B is orthogonal to G [14]. Let W_1, W_2, W_3 be an orthonormal basis of $\mathbb{R}^3$ with $W_1 = G/\|G\|$ and $W_3 = B/\|B\|$. Writing $\xi = \xi_1 W_1 + \xi_2 W_2 + \xi_3 W_3$, we have

$$\frac{d}{dt}\begin{pmatrix}\xi^1\\ \xi^2\\ \xi^3\end{pmatrix} = -K_1\begin{pmatrix}\xi^1\\ \xi^2\\ 0\end{pmatrix} - K_2\begin{pmatrix}0\\ \xi^2\\ \xi^3\end{pmatrix}. \tag{17}$$

The linearized error system is exponentially convergent around any trajectory, independently from the trajectory, and the time constants $1/K_1 > 0$ and $1/K_2 > 0$ can be chosen freely.

V. Conclusion

In this technical note, we completed the theory of [2] and [4], giving a general framework to symmetry-preserving observers when the state space is a Lie group. The observers are intrinsically and globally defined. We explained the nice properties of the error equation in two examples of [2] and [4]. In particular, we derived observers which converge around any trajectory and such that the global behavior is independent of the trajectory as well as of the time-varying inputs for a general class of systems.

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