An introduction to symmetry-preserving observers

Application to a multi-sensor data fusion tutorial example

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RÉSUMÉ. *Un observateur est un filtre qui délivre une estimation de l'état d'un système dynamique à partir de mesures imparfaites issues de divers capteurs. Dans cet article, nous considérons les systèmes non-linéaires possédant des symétries. Nous montrons que la théorie récente des observateurs invariants permet de construire automatiquement des observateurs non-linéaires candidats raisonnables à partir d'observateurs construits autour d'un seul point d'équilibre. Ces observateurs ont alors un bon comportement autour d'un continuum de points d'équilibre. Nous appliquons la méthode à la construction d'observateurs pour le problème de la fusion de données entre capteurs inertiels et capteurs de position. Les bonnes performances de ces observateurs sont illustrées par des simulations et des résultats expérimentaux.*

ABSTRACT. *An observer is a filter that provides an estimation of the state of a dynamical system from several sensors imperfect measurements. In this paper, we focus on nonlinear systems that possess symmetries. We show that the recent theory of symmetry-preserving observers allows to automatically build sensible candidate nonlinear observers from observers designed around only one particular equilibrium point. These observers behave then well around a continuum of equilibrium points. We apply the proposed methodology to design such an observer to the problem of inertial and position sensors fusion, and we illustrate the good performance of these symmetry-preserving estimators through simulation and experimental results.*

MOTS-CLÉS : *Observateur non-linéaire, Symétries, Filtre de Kalman étendu, Observateur de Luenberger, Fusion de capteurs inertiels et de position*

KEYWORDS: *Nonlinear observer, Symmetries, Extended Kalman Filter, Luenberger observer, Inertial and position sensors fusion*

1. Introduction

Observers compute an estimation of the state of a dynamical system from several sensor's measurements. An application example consists in providing an accurate estimation of the position, velocity, and orientation of a flying body, given only a sequence of observations about its position (*e.g.* from Global Position System), acceleration and angular velocity (*e.g.* from an Inertial Measurement Unit) that are corrupted by large noise, and biases. Any observer, such as the Kalman filter, uses the trusted model of the dynamics of the considered system, (equations describing the motion of the body), to drastically reduce the noise (filtering) and get an accurate estimate of the system state (location, velocity, and orientation) at each time t.

When the trusted model of the dynamics is linear, the Luenberger observer and the Kalman filter provide powerful tools to estimate the state of the system despite the presence of noise. Under the assumption of a dynamical system perturbed by (known) white Gaussian noises, the Kalman filter is optimal in the sense that it minimizes the mean-square error at each time t. One of Kalman's discovery is that, in the deterministic case, the least-square estimate is minimized by a recursive filter that is the Kalman filter, and thus can be computed in real time, as an observer. In fact, the Kalman filter is an observer.

When the underlying dynamical system model is not linear, few methods to design observers exist, except for some specific nonlinearities. To cite a few, when the systems are linearizable by output injection, standard linear techniques can be applied after suitable transformations (Hammouri *et al.*, 1992; Krener *et al.*, 1983). Some systems can be written in appropriate coordinates so that the state error evolves linearly and can thus be made to decay exponentially (Krener *et al.*, 1985). For systems possessing an observability normal form with a triangular structure in some coordinates, high-gain observers are well-suited, see (Gauthier *et al.*, 2001) and more generally (Hammouri *et al.*, 2003). Recently there has been some advances on observers where the state of the estimator does not necessarily coincide with the system's state (Andrieu *et al.*, 2006; Astolfi *et al.*, 2010). There also exists drastically different methods for state or parameter estimation such as (Fliess *et al.*, 2008).

Even when non-linear techniques do not apply, and the construction of a non-linear observer to accurately estimate the state from measurements seems difficult, the theory of linear observers can be helpful. Indeed linear differential equations play a central role in the theory of ordinary differential equations, as it is well known that any differentiable function is well approximated by a linear function around any point. Thus several “nonlinear" observers are based on a *linearization* of the system, where small errors between the estimated state and the true state (estimation error) are approximated by their first-order Taylor expansion. This leads to a linear estimation error equation to which the theory of Luenberger observers or Kalman filters can be applied. This idea is at the core of the popular extended Kalman filter (EKF), which has been proved to have some convergence properties for non-linear systems (Boutayeb *et al.*, 1997). In particular, it is well known that around an equilibrium point, the stability of any system can be generally studied thanks to the linear approximation of the equations of the system (Hartman-Grobman theorem). Thus, around such a point, one can always build a Luenberger observer or a Kalman filter having desirable properties.

The notion of transformation group is one of the most fundamental and simplest notion in mathematics. Indeed the brain tends to think in terms of invariants to group transformations (Arnold, 1974). For example, in control engineering it is usual to pilot an aircraft with a joystick controlling the velocity rather than the position. Indeed the velocity is invariant to the action of translations in space, and it is easier to deal with a control that does not depend on the special location where the plane is at. In this paper we consider a example of a system having symmetries : a GPS-aided navigation example. The equations of the aircraft are invariant to the group action of translations but also to the action of rotations because of the Galilean invariances. Around any equilibrium point one can build a Luenberger observer or a Kalman filter. Thanks to the Galilean invariances, this observer can be extended around a whole set of points, corresponding to the translated and rotated equilibrium point. Such an extension seems very natural. The resulting extended observer has a (surprising at first sight) nonlinear structure.

In the authors' opinion, this example allows to present the recent theory of symmetry-preserving (Bonnabel *et al.*, 2009a) in a tutorial way. Moreover it allows to advocate the use of symmetry-preserving observers as a mere natural extension of linear observers designed around selected equilibrium points. The resulting symmetry-preserving observers are sensible nonlinear observers that are well behaved around a large amount of equilibrium points, or even trajectories.

The theory of symmetry-preserving observers have been developed in (Bonnabel *et al.*, 2008; Bonnabel *et al.*, 2009a; Bonnabel, 2007; Bonnabel *et al.*, 2009b), with related approaches in (Maithripala *et al.*, 2005; Mahony *et al.*, 2008; Lagemann C. *et al.*, 2008), and previous attempts in (Brockett, 1980; Marcus, 1984; Krener, 1986) to introduce geometry in the problem of nonlinear filtering. Such nonlinear observers to the attitude estimation problem have been designed and experimentally validated, either using only inertial and magnetic sensors in (Martin *et al.*, 2007; Martin *et al.*, 2010), or with some additional position and velocity sensors in (Martin *et al.*, 2008b; Martin *et al.*, 2008a) (typically with measurements provided by GPS or barometer). In the previously cited works, the proposed symmetry-preserving observer has been implemented in real-time on a cheap microcontroller, to illustrate both its simplicity, and its nice convergence properties around a large set of trajectory.

The novelty of this paper lies in the chosen examples, that have never been published, and in the use of the theory of symmetry-preserving observers to design nonlinear extensions of linear observers around an equilibrium point. Though we have no proof that the resulting observers have nice global properties, we believe that this natural extension of linear observers based on the symmetries of the system has potential for broad applications.

The global nice behavior of the symmetry-preserving observers is well illustrated by experimental results of Section 5. Indeed, the observer provides an estimation of the position, velocity and orientation of a body flying in a vertical plane, from acceleration and angular velocity measurements corrupted by a great deal of noise and biases, and measurements of the position with an extremely low measurement update rate (every 1.5s). After designing as usual a linear observer around an equilibrium point, a symmetry-preserving observer is naturally derived from this linear observer, using the method described in this paper. While both filter give similar results around equilibrium points, the symmetry-preserving observer behaves much better if the system follows any trajectory.

In Section 2, we first recall the main definitions and results on linear observers. Next in Section 3, a basic example allows us to naturally derive a symmetry-preserving nonlinear observer from equations of linear observer. The general method to design such observers is then given in Section 4. Finally in Section 5, the nice global behavior of symmetry-preserving observers is illustrated through simulation and experimental results considering a GPS/IMU sensors fusion system.

We would like to thank especially Philippe Martin and Pierre Rouchon with whom the theory of symmetry-preserving observers was elaborated. In particular, the example in this paper is related to several discussions we had with them.

2. Linear observers

We recall that observers are meant to compute an estimation of the state of a dynamical system from several sensor measurements. Let $x \in \mathbb{R}^n$ denote the state of the system, $u \in \mathbb{R}^m$ be the inputs (a set of m known scalar variables such as controls, constant parameters, etc.). We assume the sensors provide measurements $y \in \mathbb{R}^p$ that can be expressed as a function of the state and the inputs. When the underlying dynamical model is a linear differential equation, and the output is a linear function as well, the system can be written

$$\frac{d}{dt}x = Ax + Bu, \qquad y = Cx + Du. \qquad [1]$$

For example, consider a vehicle equipped with a GPS and an accelerometer, moving on a straight horizontal line. Let $P \in \mathbb{R}$ be its position, $V \in \mathbb{R}$ its velocity, $a \in \mathbb{R}$ its acceleration, measured by the accelerometer. The state is $x = (P, V)^T \in \mathbb{R}^2$, the input is the accelerometer measurement $u = a$, and the output is the position measurement P of the GPS. The equations of the system write :

$$\frac{d}{dt}P = V, \qquad \frac{d}{dt}V = a, \qquad y = P. \qquad [2]$$

The aim of an observer (such as the Kalman filter) is to provide an estimation of the state, here especially the velocity that is not measured by any sensor. On one hand, as the frequency of the GPS measurement is low (4Hz - 1Hz) differentiating this measurement to identify the velocity would lead to degraded performances. On the other hand, solely integrating the accelerometer measurements (measurements refresh rate usually around 100Hz) without taking into account the GPS measurement would yield velocity errors as 1) the initial velocity is not known, 2) the accelerometer can be biased, which would lead to growing velocity estimation error.

2.1. *Luenberger observer and Kalman filter*

Consider again the linear system [1], a Luenberger observer writes

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu - L \cdot (C\hat{x} + Du - y), \qquad [3]$$

where $\hat{x}$ is the estimated state, and L is a gain matrix that can be freely chosen. We see that the observer consists in a copy of the system dynamics $A\hat{x} + Bu$, plus a correction term $L(C\hat{x} + Du - y)$ which compares the estimated output $\hat{y} = C\hat{x} + Du$ to the measured output y. The form [3] is very natural. Indeed on one hand, when the observer's output and the system output coincide on a time interval, the observer is updated via the dynamical model of the system, *i.e.*, we let the observer evolve as the true system. On the other hand, when the estimation does not coincide with the data collected from the system (the output measurement), the observer's dynamics must be corrected to reduce the estimation error. This is done adding the correction term $L(C\hat{x} + Du - y)$ which qualitatively compares the predicted output and the measured output.

One important issue is the choice (or "tuning") of the matrix L. The Luenberger observer is based on a choice of a constant matrix L, whereas in the Kalman filter L depends on the time and is updated solving a Ricatti equation. However, in both cases the observer has the form [3] with L constant or not. Let $\tilde{x} = \hat{x} - x$ be the estimation error, and let us compute the differential equation satisfied by the error. We have

$$\frac{d}{dt}\tilde{x} = (A + LC)\tilde{x}. \qquad [4]$$

As the goal of the observer is to find an estimate of x, we want $\tilde{x}$ to go to zero. When the system is observable, one can always find L such that $\tilde{x}$ asymptotically exponentially goes to zero, and the negative real part of the eigenvalues of $A + LC$ can be freely assigned. We see that the theory is particularly simple as the error equation [4] is *autonomous*, *i.e.* it does not depend on the trajectory followed by the system. In particular, the input term u has vanished in [4]. The well-known separation principle stems from this fact.

In the Kalman filter, the state equation and the output measurement are assumed to be perturbed by white Gaussian noises. If we let two positive definite matrices M and N denote the covariance matrices of the model noise and output noise, the Kalman filter equations in continuous time write

$$L = PC^T N, \qquad \frac{d}{dt}P = AP + PA^T + M^{-1} - PC^T NCP.$$

As M and N must be defined by the user, they can be viewed as tuning matrices. The Kalman filter is thus very useful when the noise covariance is known. On the other hand when nothing is known about the noise (matrices M, N) a Luenberger observer can be easier to tune. It would be erroneous to think that the Kalman filter is a much better filter than the Luenberger observer as there are deep rooted connections between both. Indeed, for any linear stationary system, and any Luenberger gain matrix L achieving exponential convergence of the Luenberger observer, there exists noise matrices M and N such that the Kalman gain asymptotically tends to L. In this case the Kalman filter asymptotically behaves just as a Luenberger observer. Finally, the main drawback of the Kalman filter is its computational cost, that makes it hard to implement on small micro-controllers with limited computational power (as those embedded on small Unmanned Aerial Vehicles-UAV's).

2.2. *A simple example*

Consider again a vehicle equipped with a GPS and accelerometers, moving on a straight horizontal line. The system equations 2 can be rewritten in the form [1] with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.$$

Choosing $L = (L_1 \; L_2)^T$, a Luenberger observer writes :

$$\frac{d}{dt}\hat{P} = \hat{V} - L_1(\hat{P} - P), \qquad \frac{d}{dt}\hat{V} = a - L_2(\hat{P} - P)$$

The estimation error equation is

$$\frac{d}{dt}\tilde{P} = \tilde{V} - L_1\tilde{P}, \qquad \frac{d}{dt}\tilde{V} = -L_2\tilde{P},$$

and eliminating $\tilde{V}$ we see that the estimation error converges exponentially to 0 for any $L_1, L_2 > 0$. The final choice of L_1, L_2 depends on the tradeoff between measurement noise and modeling errors (encoded in the matrices M and N in the Kalman filter theory). Since GPS position measurement is only available approximately every second, it makes no sense for the error equation to converge too quickly to 0, choosing gains L_1, L_2 too large. On the other hand, choosing L_1, L_2 too small leads to an error equation that converges too slowly. This tradeoff is illustrated by experimental results of Section 5.

3. Natural derivation of linear observer for a nonlinear system

Consider a general nonlinear system

$$\frac{d}{dt}x = f(x, u), \qquad y = h(x, u), \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input, and $y \in \mathbb{R}^p$ the output. Mimicking the linear case, a class of popular nonlinear observers writes

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u) - L(\hat{x}, y, t) \cdot \big(h(\hat{x}, u) - y(t)\big),$$

where the gain matrix can depend on the variables $\hat{x}, y, t$. The error equation can still be computed, but as the system is nonlinear, it does not necessarily lead to an appropriate gain matrix L. Indeed let $\tilde{x}$ be the error, we have

$$\frac{d}{dt}\tilde{x} = f(\hat{x}, u(t)) - f(x, u(t)) - L(\hat{x}, y(t), t) \cdot \big(h(\hat{x}, u(t)) - y(t)\big). \qquad [6]$$

The error equation is no longer autonomous, and the problem of finding L such that $\tilde{x}$ goes asymptotically to zero can not be solved in the general case.

3.1. *Design around an equilibrium point*

It is a central idea of analysis that any differentiable function is well approximated by a linear function around any point. Consider an equilibrium point $(\bar{x}, \bar{u}, \bar{y})$ characterized by $f(\bar{x}, \bar{u}) = 0$ and $\bar{y} = h(\bar{x}, \bar{u})$. Let $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$, and $\delta y = y - \bar{y}$. Around the equilibrium point, we have up to second order terms in $\delta x, \delta u, \delta y$:

$$\frac{d}{dt}\delta x = A\delta x + B\delta u, \qquad \delta y = C\delta x + D\delta u,$$

where

$$A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}),\ B = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}),\ C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}),\ D = \frac{\partial h}{\partial u}(\bar{x}, \bar{u}).$$

If the linearized system is observable, consider the following observer

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u) - L(\hat{y} - y) \qquad [7]$$

where we have chosen the observer gain matrix L such that $A + LC$ is a stable matrix. It achieves local exponential convergence around the equilibrium. Indeed, let $\tilde{x} = \hat{x} - x = \delta\hat{x} - \delta x$. The error equation around $\bar{x}, \bar{u}, \bar{y}$ is well approximated by the following linear equation :

$$\frac{d}{dt}\tilde{x} = (A + LC)\tilde{x},$$

which is the same as [4]. The most popular observer designed for a nonlinear system is the Extended Kalman Filter (EKF). The principle is to linearize the system around the estimated trajectory, build a Kalman filter for the linear model, and implement it on the nonlinear system. The EKF has the form [7], with the gain matrix computed the following way :

$$A = \frac{\partial f}{\partial x}(\hat{x}, u) \qquad L = PC^T N$$

$$C = \frac{\partial h}{\partial x}(\hat{x}, u) \qquad \frac{d}{dt}P = AP + PA^T + M^{-1} - PC^T NCP.$$

Around an equilibrium point, the system can be approximated by the linearized system around this point, and the Extended Kalman Filter boils down to a linear Kalman filter on the linearized model. Far from equilibriums (*i.e.* when δx is not small), there is no guarantee that either the Luenberger observer [7] or the EKF achieve convergence of the estimation error to zero, as nothing can be concluded from the error equation [6]. Moreover, the EKF is no longer the optimal estimated state (or least squares estimate) when the model is nonlinear. Thus, far from equilibriums not much can be said about the behavior of the observer in the general nonlinear case.

3.2. *A natural extension of the linear design : tutorial example*

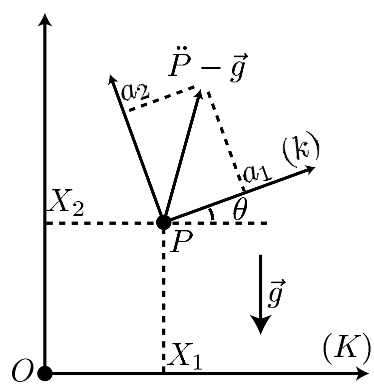

Figure 1. *A flying object in a vertical plane*

The considered example is a flying object (*e.g.* a small unmanned aerial vehicle) moving in a vertical plane. This system moves in two dimensions, and is equipped with a GPS and IMU (accelerometers and gyroscopes) (see (Rouchon, 2010)). Moreover for technical reasons (observability of the biases), we assume it is also equipped with a camera. The system's state is its position $(X_1, X_2)^T \in \mathbb{R}^2$, its velocity $(V_1, V_2)^T \in \mathbb{R}^2$, and its orientation $\theta \in \mathbb{S}^2$. The GPS measures the position, and the gyroscopes measure the angular velocity ω. The accelerometers measure, in the body frame (a frame attached to the flying object), the acceleration induced by all external forces *but* gravity. Indeed, accelerometers are composed by small loads that are sensitive to accelerations, but also to gravity. As a result, if the IMU is static, the vertical accelerometer indicates an upward acceleration of $9.81 m/s^2$. To be more realistic, we suppose the accelerometers measurements are biased, and to get a little simpler system, we neglect the gyroscope bias. Two constant biases b_1 and b_2 are thus added to the accelerometers measurements a_1 and a_2. Finally we suppose that the flying object is equipped with a camera capable of tracking a fixed horizontal line. Thus, it measures the angle θ. Noise also corrupts all the measurements ; it can dealt with indirectly through the tuning of the observer gains (*e.g.* Luenberger observer), and directly considering noise in the model (*e.g.* EKF). Finally the equations of the system write

$$\begin{aligned} \frac{d^2}{dt^2}X_1 &= (a_1(t)+b_1)\cos(\theta) - (a_2(t)+b_2)\sin(\theta) \\ \frac{d^2}{dt^2}X_2 &= (a_1(t)+b_1)\sin(\theta) + (a_2(t)+b_2)\cos(\theta) - g \\ \frac{d}{dt}\theta &= \omega(t) \\ y &= (X_1, X_2, \theta)^T. \end{aligned} \qquad [8]$$

This dynamical model is obviously nonlinear because of the transcendental functions sine and cosine. The goal of an observer is to filter the several measurements, identify the biases, and deliver a good estimation of the velocity, which is not measured.

3.2.1. *Design around* $\theta = 0$

To linearize this model, we are going to suppose that θ is small, and also that the flight is quasi-static, *i.e.*, the kinematic acceleration of the body is small compared to gravity, leading to the assumption $|a_1| \ll g$ and $|a_2 - g| \ll g$. Indeed the equilibrium point characterized by $X_1 = 0, X_2 = 0, \theta = 0$ in [8] is such that $a_1 = 0, a_2 = g$. Up to second order terms the model is

$$\frac{d^2}{dt^2}X_1 = a_1(t) + b_1 - g\theta, \qquad \frac{d}{dt}b_1 = 0, \qquad \frac{d}{dt}\theta = \omega(t) \qquad [9]$$

$$\frac{d^2}{dt^2}X_2 = a_2(t) + b_2 - g, \qquad \frac{d}{dt}b_2 = 0 \qquad [10]$$

and it is a good approximation of [8]. We used a standard trick which consists in including the biases b_1 and b_2 in the state, so that the observer will allow to estimate the biases, as it provides an estimation of the full state. We see there is a coupling between X_1, V_1, b_1 and θ. Thus the linearized system can be decomposed in two independent subsystems [9] and [10]. A Luenberger observer (or a Kalman filter) can be built on each subsystem. The resulting observer writes :

$$\begin{aligned}
\frac{d}{dt}\hat{X}_1 &= \hat{V}_1 - L^X_{X_1}(\hat{X}_1 - y_1) + L^\theta_{X_1}(\hat{\theta} - y_3), \\
\frac{d}{dt}\hat{V}_1 &= a_1(t) + \hat{b}_1 - g\hat{\theta} - L^X_{V_1}(\hat{X}_1 - y_1) + L^\theta_{V_1}(\hat{\theta} - y_3) \\
\frac{d}{dt}\hat{\theta} &= \omega(t) - L^X_\theta(\hat{X}_1 - y_1) - L^\theta_\theta(\hat{\theta} - y_3) \\
\frac{d}{dt}\hat{b}_1 &= -L^X_{b_1}(\hat{X}_1 - y_1) - L^\theta_{b_1}(\hat{\theta} - y_3) \\
\frac{d}{dt}\hat{X}_2 &= \hat{V}_2 - L_{X_2}(\hat{X}_2 - y_2), \\
\frac{d}{dt}\hat{V}_2 &= a_2(t) + \hat{b}_2 - g - L_{V_2}(\hat{X}_2 - y_2) \\
\frac{d}{dt}\hat{b}_2 &= -L_{b_2}(\hat{X}_2 - y_2).
\end{aligned}$$

On each subsystem, the gain matrices can be chosen to freely assign the real part of the eigenvalues of the error equation. Thus around $\theta = 0$, in quasi-static flight, it is easy to design an asymptotically converging Luenberger observer (or a Kalman filter).

3.2.2. *Design around* $\theta = \pi/2$

Consider now a quasi-static flight such that θ is close to $\pi/2$, $|a_2| \ll g$ and $|a_1 - g| \ll g$. Up to second order terms the model is

$$\frac{d^2}{dt^2}X_1 = -a_2(t) - b_2 + g(\pi/2 - \theta), \qquad \frac{d}{dt}b_2 = 0, \qquad \frac{d}{dt}\theta = \omega(t) \quad [11]$$

$$\frac{d^2}{dt^2}X_2 = a_1(t) + b_1 - g, \qquad \frac{d}{dt}b_1 = 0 \quad [12]$$

and it is a good approximation of [8]. The system can be decomposed in two independent subsystems [11] and [12]. Thus the equations of any Luenberger observer (or Kalman filter) for this observer will be the same as above but for the biases equations, as now b_2 belongs to the first subsystem, whereas b_1 belongs to the second one, contrarily to the previous case (linearization around $\theta = 0$). A Luenberger observer can be designed on the linear system. Only writing the biases equations (to save space) it writes :

$$\begin{aligned}
\frac{d}{dt}\hat{b}_1 &= L_{b_1}(\hat{X}_2 - y_2) \\
\frac{d}{dt}\hat{b}_2 &= -L^X_{b_2}(\hat{X}_1 - y_1) - L^\theta_{b_2}(\hat{\theta} - y_3)
\end{aligned}$$

as the roles of a_1 and a_2 have been exchanged (up to a sign a multiplication by ± 1).

3.2.3. *Design around any* $\theta = \bar{\theta}$

Now consider a linearization around any constant value $\bar{\theta}$. We can intuit from the two previous linearizations that there would be some benefit in designing gains for the biases equations compatible with rotations of the body. A sensible way to “interpolate" between the linearizations around $\theta = 0$ and $\theta = \pi/2$, for arbitrary θ, would be (only writing the biases equations) :

$$\frac{d}{dt}\hat{b}_1 = -L^{X_1}(\hat{X}_1 - y_1)\cos(\hat{\theta}) + L^{X_2}(\hat{X}_2 - y_2)\sin(\hat{\theta}) - L^{\theta}(\hat{\theta} - y_3)\cos(\hat{\theta})$$
$$\frac{d}{dt}\hat{b}_2 = -L^{X_1}(\hat{X}_1 - y_1)\sin(\hat{\theta}) - L^{X_2}(\hat{X}_2 - y_2)\cos(\hat{\theta}) - L^{\theta}(\hat{\theta} - y_3)\sin(\hat{\theta}).$$

[13]

Indeed, around any constant value of θ, along with the quasi-static assumption, the dynamics of the true system decouples in two subsystems (horizontal and vertical subsystems). So does the observer (along with the previous equations for the rest of the state). It is locally converging around any $\bar{\theta}$. In particular, for $\theta = 0$ and $\theta = \pi/2$ we recover the observers above (with an additional constraint on the gains, as the tuning does not depend on the orientation of the body frame). Such an observer is called a symmetry-preserving observer (Bonnabel *et al.*, 2008). Its structure mimics the system’s structure. We see that if the tuning is correctly done around $\theta = 0$ (using a pole assignment or a Kalman filter) it can be *automatically extended* around any constant θ via a rotation of the gains. The resulting observer is well behaved locally, and overall has a nonlinear form [13]. In fact, this observer can be directly designed using formula [15] from the theory of symmetry-preserving observers.

4. Symmetry-preserving observers

In the previous example, we saw there was a very natural way to extend the design around a point to some other points. Indeed, we considered several configurations of the orientation of the body, each of them being linked to the other by a rotation. We figured out that rotating the correction terms of the observer, just the same way as the body rotates, would be beneficial as it allows to extend the tuning around one point to any other of those points. In fact, the design is based on the fact that rotating the axes of the body does not change the equations of the system, so it should not change much the observer design. Indeed a rotation of the axes of the body does not necessarily correspond to an actual rotation of the body itself, but it can be viewed as another choice of the body frame. And the equations of the body motion do not depend on any non-trivial choice of orientation of the body frame : they are *invariant* to a body frame rotation. The mere statement that the observer must also be invariant to a body frame rotation automatically yields correction terms of the form [13].

4.1. *How can a model be symmetrical ?*

Essentially, a thing is symmetrical if there is an operation we can do to it so that after the operation it looks the same (Feynman, 1979). For instance a sphere is symmetrical as it looks the same whatever rotation it is subject to. An object is thus symmetrical if it is invariant to some transformations. So the question is what operation can we do to a physical experiment, or a dynamical system, and leave the result the same ? Most famous operations under which various physical phenomena are invariant are : translation in space, translation in time, rotation through a fixed angle, reflection in space, interchange of identical atoms etc.

In a more mathematical way, let G be a group, and M be a set. A group action can be defined on M if to any $g \in G$ on can associate a diffeomorphic transformation $\phi_g : M \to M$ such that $\phi_{gh} = \phi_g \circ \phi_h$, and $(\phi_g)^{-1} = \phi_{g^{-1}}$, *i.e.*, the group multiplication corresponds to the transformation composition, and the reciprocal elements correspond to reciprocal transformations. Consider the general system [5]. Consider also the local group of transformations on $\mathcal{X} \times \mathcal{U}$ defined for any x, u, g by

$$(X, U) = \big(\varphi_g(x), \psi_g(u)\big), \qquad [14]$$

where φ_g and ψ_g are local diffeomorphisms.

Definition 1 *The system* $\frac{d}{dt}x = f(x, u)$ *is* G-invariant *if* $\frac{d}{dt}X = f(X, U)$*, for all* g, x, u *where* X, U *are defined by* [14],

i.e., the system remains unchanged under the transformation [14]. The property also reads $f\big(\varphi_g(x), \psi_g(u)\big) = D\varphi_g(x) \cdot f(x, u)$. We understand from this definition, that u can denote the control variables as usual, but it also denotes every feature of the environment that makes the system not behave the same way after it has been transformed (via φ_g). The action of ψ_g is meant to allow some features of the environment to be also moved over.

For example, consider the model above [8]. It is invariant to the action of rotations. Indeed let $p = (X_1, X_2)^T$ be the position vector, $v = (V_1, V_2)$ be the velocity vector, $a = (a_1, a_2)$ the accelerometer measurement, $b = (b_1, b_2)$ the biases, $\mathbf{g}_{grav}$ the earth gravity vector, and let $R_\theta \in SO(2)$ denote the rotation matrix of fixed angle θ. The model [8] can be written

$$\frac{d}{dt}p = v, \qquad \frac{d}{dt}v = R_\theta(a + b) + \mathbf{g}_{grav}, \qquad \frac{d}{dt}\theta = \omega, \qquad \frac{d}{dt}b = 0$$

Let $x = (p, v, \theta, b)$ be the state, and $u = (a, \omega)$ be the input, let the group action of $G = SO(2)$ correspond to rotations in the body frame, *i.e.*, for any angle $g \in \mathbb{S}^1$:

$$(X, U) = \big(\varphi_g(x), \psi_g(u)\big) = \big((p, v, \theta - g, R_g b), (R_g a, \omega)\big)$$

Obviously we have $\frac{d}{dt}X = f(X, U)$ proving that the model is invariant to the action of $SO(2)$ and thus possesses symmetries. Symmetries are often linked to invariance to a change of coordinates or units, and here the intuition for choosing this group stems from the fact that the dynamics does not depend on any non-trivial choice of orientation of the body frame axes (nor on the orientation of earth-fixed frame axes, as we will see later).

4.2. *Symmetry-preserving observer design*

Now that we have explained what a symmetry is, and that we have found symmetries in the example, we can wonder how we should transform the usual formulas for popular observers [6] so that the observer has the same symmetries as the system, and what would be the benefit of this transformation. In this subsection we recall the basic definitions and results of (Bonnabel *et al.*, 2008).

Definition 2 *The observer* [6]

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u(t)) - L(\hat{x}, y(t), t) \cdot \big(h(\hat{x}, u(t)) - y(t)\big)$$

is G-invariant *or "symmetry-preserving" if for all* $g, x, \hat{x}, u, y$*, it is invariant to the transformation*

$$(x, \hat{x}, u) \mapsto (X, \hat{X}, U) = (\varphi_g(x), \varphi_g(\hat{x}), \psi_g(u))$$

i.e.

$$\frac{d}{dt}\hat{X} = f(\hat{X}, U(t)) - L(\hat{X}, Y(t), t) \cdot \big(h(\hat{X}, U(t)) - Y(t))\big)$$

where $Y = h(X, U)$.

In this case, the structure of the observer mimics the nonlinear structure of the system. Let us recall how to build such observers (see (Bonnabel *et al.*, 2008) for more details). To do so, we need the output to be equivariant :

Definition 3 *The output is equivariant if for all* g, x, u *there exists a diffeomorphism* ρ_g *on the output space (group action) such that*

$$h(\varphi_g(x), \psi_g(u)) = \rho_g(h(x, u))$$

We will systematically assume the output is equivariant. In the examples considered here the output is a part of the state variables and it is thus obviously equivariant. We are now going to explain how to define symmetry-preserving candidate observers. The first ingredient is an invariant output error, instead of the usual linear output error $\hat{y} - y$:

Definition 4 *The smooth map* $(\hat{x}, u, y) \mapsto E(\hat{x}, u, y) \in \mathbb{R}^p$ *is an* invariant output error *if*

– *the map* $y \mapsto E(\hat{x}, u, y)$ *is invertible for all* $\hat{x}, u$
– $E\big(\hat{x}, u, h(\hat{x}, u)\big) = 0$ *for all* $\hat{x}, u$
– $E\big(\varphi_g(\hat{x}), \psi_g(u), \rho_g(y)\big) = E(\hat{x}, u, y)$ *for all* $\hat{x}, u, y$

The construction of such errors is based on Cartan moving frame method that we will not detail in this paper (see (Bonnabel *et al.*, 2008; Aghannan *et al.*, 2002) for more details). Indeed, when the transformation group can be identified to a subset of the state, the construction of invariant output errors is particularly simple. Suppose that the state variable can be written $x = (x_1, x_2)$ with $x_2 \in G$. An invariant output error is given by

$$E(\hat{x}, u, y) = \rho_{\hat{x}_2^{-1}}(\hat{y}) - \rho_{\hat{x}_2^{-1}}(y)$$

where $\hat{y} = h(\hat{x}, u)$. In the example above, the output is $h(p, v, \theta, b) = (p, \theta)$, with $\theta \in G$. We have

$$\rho_g(p, \theta) = (p, \theta - g)$$

as $\varphi_g(p, v, \theta, b) = (p, v, \theta - g, R_g b)$. The usual output error $\hat{y} - y = (\hat{p} - p, \hat{\theta} - \theta)$ is thus the invariant output above as here $x_2 = \theta$, and thus $\rho_{\hat{x}_2^{-1}}(\hat{y}) = (\hat{p}, 0)$, and $\rho_{\hat{x}_2^{-1}}(y) = (p, \theta - \hat{\theta})$.

A few more definitions are needed. I is called a complete set of scalar invariants and verifies $I(\varphi_g(\hat{x}), \psi_g(u)) = I(\hat{x}, u)$ for any $g \in G$. Once again, it can be built thanks to the moving frame method. Suppose again that the state can be written $x = (x_1, x_2)$ with $x_2 \in G$. Then a complete set of invariants is given by

$$I(\hat{x}, u) = (\varphi_{\hat{x}_2^{-1}}(\hat{x}), \psi_{\hat{x}_2^{-1}}(u))$$

Definition 5 *A vector field* w *on* $\mathcal{X}$ *is said to be G-invariant if the system* $\frac{d}{dt}x = w(x)$ *is invariant. This means for any* g *if we let* $X(t) = \varphi_g(x(t))$ *we have* $\frac{d}{dt}X(t) = w(X(t))$ *for all* t.

Definition 6 *An invariant frame* $(w_1, ..., w_n)$ *on* $\mathcal{X}$ *is a set of* n *linearly point-wise independent G-invariant vector fields, i.e* $(w_1(x), ..., w_n(x))$ *is a basis of the tangent space to* $\mathcal{X}$ *at* x.

Once again such a frame can be built thanks to the moving frame method. In particular suppose the state is $x = (x_1, x_2)$ with $x_2 \in G$. Then an invariant frame is made of n vector fields whose components are the columns of the inverse of the Jacobian $(D\varphi_{x_2{}^{-1}}(x)))^{-1}$ composed by the partial derivatives of the components $\varphi_g(\hat{x})$ with respect to the components of $\hat{x}$. In the example, θ is a state variable, and it can be identified to an element of the transformation group. Thus $x_2 = \theta$ and the reciprocal of $\hat{\theta}$ for the group multiplication law is $-\theta$. As $\varphi_g(p,v,\theta,b) = (p,v,\theta-g,R_g b)$, an invariant frame is given by the columns of the matrix

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&\cos\theta&\sin\theta\\0&0&0&0&0&-\sin\theta&\cos\theta\end{pmatrix}^{-1} = \begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&\cos\theta&-\sin\theta\\0&0&0&0&0&\sin\theta&\cos\theta\end{pmatrix}$$

Finally, a sufficient condition for the system $\frac{d}{dt}\hat{x} = F(\hat{x},u,y)$ to be a G-invariant pre-observer for the G-invariant system $\frac{d}{dt}x = f(x,u)$ is (Bonnabel *et al.*, 2008) :

$$F(\hat{x},u,y) = f(\hat{x},u) + \sum_{i=1}^{n} \mathcal{L}_i\big(I(\hat{x},u), E(\hat{x},u,y)\big) w_i(\hat{x}) \qquad [15]$$

where E is an invariant output error, $I(\hat{x},u)$ is a full-rank invariant function, the $\mathcal{L}_i$'s are smooth functions such that for all $\hat{x}$, $\mathcal{L}_i\big(I(\hat{x},u),0\big) = 0$, and $(w_1,...,w_n)$ is an invariant frame. The gains $\mathcal{L}_i$ must be tuned in order to get some convergence properties if possible, and their magnitude should depend on the trade-off between measurement noise and convergence speed. It should be mentioned that the links between measurement noise, symmetries of the system, and tuning of the gains $\mathcal{L}_i$ is studied in (Bonnabel *et al.*, 2009b). In the example above, this form corresponds to usual linear correction terms on the variables $\hat{X}_1, \hat{V}_1, \hat{X}_2, \hat{V}_2, \hat{\theta}$ and nonlinear correction terms [13] on the biases, as the output error is the usual linear output error, and the invariant frame depends on the state variables $\hat{\theta}$, only for the biases components.

Thus, to build a symmetry-preserving observer, one needs a) an invariant output error b) an invariant frame c) a complete set of scalar invariants.

5. Example : inertial and position sensor fusion

We consider the same example as previously without the camera. We suppose that the accelerometers and gyroscope measurements are unbiased, since the existing biases can be easily estimated when the system is at rest. The goal is to estimate the position, velocity, and orientation of a flying body, using accelerometers, gyros, and position measurements from GPS. The equations write

$$\begin{aligned} \frac{d}{dt}V_1 &= a_1(t)\cos(\theta) - a_2(t)\sin(\theta) \\ \frac{d}{dt}X_1 &= V_1 \\ \frac{d}{dt}V_2 &= a_1(t)\sin(\theta) + a_2(t)\cos(\theta) - g \\ \frac{d}{dt}X_2 &= V_2 \\ \frac{d}{dt}\theta &= \omega(t) \\ y &= (X_1, X_2)^T. \end{aligned} \qquad [16]$$

The state is $x = (X_1, X_2, V_1, V_2, \theta)$, the input $u = (a_1, a_2, \mathbf{g}_{grav_1}, \mathbf{g}_{grav_2}, \omega)$ where $(\mathbf{g}_{grav_1}, \mathbf{g}_{grav_2}) = (0, -g)$ are the coordinates of the gravity vector. Indeed as the transformation corresponds to a change of orientation of the axes of the inertial frame, the gravity vector must be rotated as well in order to leave the system the same, so it must be part of the input. As the angle θ is not directly measured, the system is observable only if is not in free fall (in which case there is no way to distinguish from the sensor measurements where is up and down). This means that the system is observable only if $(a_1^2(t) + a_2^2(t))) \neq 0$ on any interval $T \subset \mathbb{R}$.

Contrarily to the previous example, we are going to consider the transformation group of rotations and translations of the inertial frame : $G = SE(2)$. To define the group action, it suffices to note that $y = (X_1, X_2)^T$ are coordinates of a vector in the inertial frame, (a_1, a_2) are coordinates of a vector of the body frame, and $(\mathbf{g}_{grav_1}, \mathbf{g}_{grav_2})$ those of a vector expressed in the inertial frame. We have

$$\varphi_g(x) = \begin{pmatrix} R_{-g_3}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ R_{-g_3}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\ \theta + g_3 \end{pmatrix}, \quad \psi_g(u) = \begin{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ R_{-g_3}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \\ u_5 \end{pmatrix}$$

$$\varrho_g(y) = \left(R_{-g_3}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right), \quad R_{g_3} = \begin{pmatrix} \cos g_3 & \sin g_3 \\ -\sin g_3 & \cos g_3 \end{pmatrix}$$

where R_{g_3} is associated to a rotation of fixed angle in the inertial frame.

5.1. *A symmetry-preserving observer*

Let us linearize the system supposing as before that θ is small and that $(u_1-u_3)^2+(u_2-u_4)^2 \ll g_{\text{grav}}^2$, $(u_3=0,\, u_4=-g_{\text{grav}})$

$$\frac{d^2}{dt^2}X_1 = u_1 + g_{\text{grav}}\theta$$
$$\frac{d^2}{dt^2}X_2 = u_2 - g_{\text{grav}},\ \frac{d}{dt}\theta = u_5$$
$$y = (X_1, X_2)$$

A standard converging Luenberger observer is

$$\frac{d}{dt}\hat{X}_1 = \hat{V}_1 - L_{X_1}(\hat{X}_1 - y_1),$$
$$\frac{d}{dt}\hat{V}_1 = a_1(t) - g\hat{\theta} - L_{V_1}(\hat{X}_1 - y_1)$$
$$\frac{d}{dt}\hat{\theta} = \omega(t) - L_\theta(\hat{X}_1 - y_1)$$
$$\frac{d}{dt}\hat{X}_2 = \hat{V}_2 - L_{X_2}(\hat{X}_2 - y_2),$$
$$\frac{d}{dt}\hat{V}_2 = a_2(t) - g - L_{V_2}(\hat{X}_2 - y_2)$$

Indeed the system is made of two independent subsystems : with $L_{X_2}, L_{V_2} > 0$, the variables $\hat{X}_2$ and $\hat{V}_2$ satisfying

$$\frac{d}{dt}\hat{X}_2 = \hat{V}_2 - L_{X_2}(\hat{X}_2 - y_2), \quad \frac{d}{dt}\hat{V}_2 = u_2 - g_{\text{grav}} - L_{V_2}(\hat{X}_2 - y_2),$$

converge towards $y_2 = X_2$ and V_2. With $L_{X_1}, L_{V_1}, L_\theta$ computed in order to get three stable eigenvalues (pole assignment), the estimated variables $\hat{X}_2$, $\hat{V}_2$ and $\hat{\theta}$ satisfying

$$\frac{d}{dt}\hat{X}_1 = \hat{V}_1 - L_{X_1}(\hat{X}_1 - y_1), \quad \frac{d}{dt}\hat{V}_1 = u_1 + g_{\text{grav}}\hat{\theta} - L_{V_1}(\hat{X}_1 - y_1),$$
$$\frac{d}{dt}\hat{\theta} = u_5 - L_\theta(\hat{X}_1 - y_1)$$

converge towards $y_1 = X_1$, V_1 and θ.

5.2. *Invariant output errors and invariant vector fields*

In this section, we build a nonlinear observer extending to arbitrary θ the structure of the observer above designed for θ small. Everything lies on the construction of invariant output errors and an invariant frame. As the state is $(X_1, X_2, V_1, V_2, \theta)$ and (X_1, X_2, θ) can be viewed as an element of the group $SE(2)$, and invariant output error and an invariant frame can be constructed using the reciprocal element of the group element $(\hat{X}_1, \hat{X}_2, \hat{\theta})$:

$$g_1 = -\hat{X}_1 \cos\hat{\theta} - \hat{X}_2 \sin\hat{\theta}, \quad g_2 = \hat{X}_1 \sin\hat{\theta} - \hat{X}_2 \cos\hat{\theta}, \quad g_3 = -\hat{\theta}.$$

Indeed suppose the state is $x = (x_1, x_2)$ with $x_2 \in G$. Then we recall an invariant ouptut error is made of the components of $\varrho_{\hat{x}_2^{-1}}(\hat{y}) - \varrho_{\hat{x}_2^{-1}}(y)$:

$$E_1 = (\hat{y}_1 - y_1)\cos\hat{\theta} + (\hat{y}_2 - y_2)\sin\hat{\theta}, \quad E_2 = -(\hat{y}_1 - y_1)\sin\hat{\theta} + (\hat{y}_2 - y_2)\cos\hat{\theta}.$$

Scalar invariants are obtained via $I(\hat{x}, u) = (\varphi_{\hat{x}_2^{-1}}(\hat{x}), \psi_{\hat{x}_2^{-1}}(u))$. Selecting the third and fourth coordinates of $\psi_{\hat{x}_2^{-1}}(u))$ yields the two following useful invariants :

$$I_1 = u_3 \cos\hat{\theta} + u_4 \sin\hat{\theta}, \quad I_2 = -u_3 \sin\hat{\theta} + u_4 \cos\hat{\theta}.$$

An invariant frame is made of n vector fields whose components are the columns of the inverse of the Jacobian $\left(D\varphi_{\hat{x}_2^-1}(x))\right)^{-1}$ composed by the partial derivatives of the components $\varphi_g(\hat{x})$ with respect to the components of $\hat{x}$:

$$\begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} & 0 & 0 & 0 \\ -\sin\hat{\theta} & \cos\hat{\theta} & 0 & 0 & 0 \\ 0 & 0 & \cos\hat{\theta} & \sin\hat{\theta} & 0 \\ 0 & 0 & -\sin\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\hat{\theta} & -\sin\hat{\theta} & 0 & 0 & 0 \\ \sin\hat{\theta} & \cos\hat{\theta} & 0 & 0 & 0 \\ 0 & 0 & \cos\hat{\theta} & -\sin\hat{\theta} & 0 \\ 0 & 0 & \sin\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.3. *A symmetry-preserving observer*

For small $|\theta|$, $E_1 \approx (\hat{X}_1 - y_1)$ and $E_2 \approx (\hat{X}_2 - y_2)$. Thus a symmetry-preserving extension of the linear observer

$$\begin{aligned} &\frac{d}{dt}\hat{X}_2 = \hat{V}_2 - L_{X_2}(\hat{X}_2 - y_2), \quad \frac{d}{dt}\hat{V}_2 = a_2 - g_{\text{grav}} - L_{V_2}(\hat{X}_2 - y_2), \\ &\frac{d}{dt}\hat{X}_1 = \hat{V}_1 - L_{X_1}(\hat{X}_1 - y_1), \quad \frac{d}{dt}\hat{V}_1 = a_1 + g_{\text{grav}}\hat{\theta} - L_{V_1}(\hat{X}_1 - y_1), \\ &\frac{d}{dt}\hat{\theta} = \omega - L_\theta(\hat{X}_1 - y_1) \end{aligned} \quad [17]$$

is given by the following nonlinear symmetry-preserving observer [18] that we will validate thanks to simulations and experimentations

$$\begin{aligned}
\frac{d}{dt}\hat{X}_1 &= \hat{V}_1 - L_{X_1}E_1\cos\hat{\theta} + L_{X_2}E_2\sin\hat{\theta} \\
\frac{d}{dt}\hat{X}_2 &= \hat{V}_2 - L_{X_1}E_1\sin\hat{\theta} - L_{X_2}E_2\cos\hat{\theta} \\
\frac{d}{dt}\hat{V}_1 &= a_1\cos\hat{\theta} - a_2\sin\hat{\theta} + u_3 - L_{V_1}E_1\cos\hat{\theta} + L_{V_2}E_2\sin\hat{\theta} \\
\frac{d}{dt}\hat{V}_2 &= a_1\sin\hat{\theta} + a_2\cos\hat{\theta} + u_4 - L_{V_1}E_1\sin\hat{\theta} - L_{V_2}E_2\cos\hat{\theta} \\
\frac{d}{dt}\hat{\theta} &= \omega - L_\theta(I_1E_2 + I_2E_1)
\end{aligned} \qquad [18]$$

as the gain can be a function of scalar invariants I. We recall $u_1 = a_1$, $u_2 = a_2$, $u_3 = 0$, $u_4 = -g_{\text{grav}}$, $u_5 = \omega$ and

$$\begin{aligned}
E_1 &= (\hat{X}_1 - y_1)\cos\hat{\theta} + (\hat{X}_2 - y_2)\sin\hat{\theta} \\
E_2 &= -(\hat{X}_1 - y_1)\sin\hat{\theta} + (\hat{X}_2 - y_2)\cos\hat{\theta} \\
I_1 &= u_3\cos\hat{\theta} + u_4\sin\hat{\theta}, \quad I_2 = -u_3\sin\hat{\theta} + u_4\cos\hat{\theta}.
\end{aligned}$$

Note that such an extension can also be applied to a Kalman filter. Indeed the gains $L_{X_1}, L_{X_2}, L_{V_1}, L_{V_2}, L_{\theta_1}$ can be computed building a Kalman filter on the linear model. Such a Kalman filter can also be extended invariantly (see (Bonnabel *et al.*, 2009b; Bonnabel, 2007)).

Remark 1 *Note that the correction terms have a physical meaning. The output error is expressed in the body frame (invariant output error). Then, the correction terms can be interpreted as a direct correction of the acceleros measurements, that are expressed in the body frame.*

5.4. *Simulation results*

We first illustrate on simulations the behavior of the proposed symmetry-preserving filter [18]. Three kinds of sensors are used : two biaxial accelerometers measure a_{1m}, a_{2m} ; one gyroscope measures ω_m ; and one position sensor (*e.g.* GPS or camera) provides X_{1m}, X_{2m}. To validate the filtering algorithm for a realistic system, some sensors imperfections must be considered. As previously mentioned, the measurements are supposed unbiased : this assumption is technologically relevant, since the existing biases are very slowly time-varying constant and thus can be easily

Tableau 1. *Noise variance for inertial and position sensors (simulation)*

Noise	w_1, w_2	w_ω	v_1, v_2
Variance	1 $(m/s^2)^2$	0.3 s^{-2}	0.5 m^2

estimated when the system is at rest. However, we assume that all measurements are corrupted by gaussian noise, which is a model customary used. We therefore consider

$$a_{1m} = a_1 + w_1 \qquad a_{2m} = a_2 + w_2 \qquad \omega_m = \omega + w_\omega$$
$$X_{1m} = X_1 + v_1 \qquad X_{2m} = X_2 + v_2,$$

where w_i, v_i are independent normally distributed random scalars with mean 0 and variance given in Table 1. Notice that the considered variances for gyroscope and accelerometers measurements correspond to noise of very inaccurate inertial sensors, as can be seen in Figure 2. The system follows a trajectory quite representative of a small unmanned aerial vehicle flight. We choose $L_{X_1} = 5.8$, $L_{X_2} = 8.5$, $L_{V_1} = 9.5$, $L_{V_2} = 9$, $L_\theta = 3.1e - 2$, where the gains are defined in [18]. To illustrate the large domain of convergence of the observer, we initialize the states very far from their true values :

$$\hat{\theta}(0) - \theta(0) = -150^\circ$$
$$\hat{V}_1(0) - V_1(0) = \hat{V}_2(0) - V_2(0) = -5m/s$$
$$\hat{X}_1(0) - X_1(0) = \hat{X}_2(0) - X_2(0) = 10m.$$

We see on Figures 3–5 that the observer converges well even if the states are initialized far from their true values and the system moves quite fast. Though no proof of convergence is given, the domain of convergence of the observer [18] then seems to be very large. Once the observer has converged, the estimations remain very close to the true state.

5.5. *Experimental results*

We now validate the good behavior of the nonlinear observer [18] through experimental results. We also highlight the importance of the nonlinear "symmetry-preserving" aspect in the correction terms of [18], by comparing the estimations with those provided by an observer with linear correction terms (observer which consists in a copy of the dynamics, plus the correction terms of [17]). The experimental setup is shown in Figure 6. An Inertial Measurement Unit (IMU) Crossbow VG is moved in any direction along a wall. Since we do not have any device that estimates the current position of the system, a "position sensor" is simulated by drawing waypoints on the wall at known positions, and by reaching one waypoint every 1.5 second. We save the

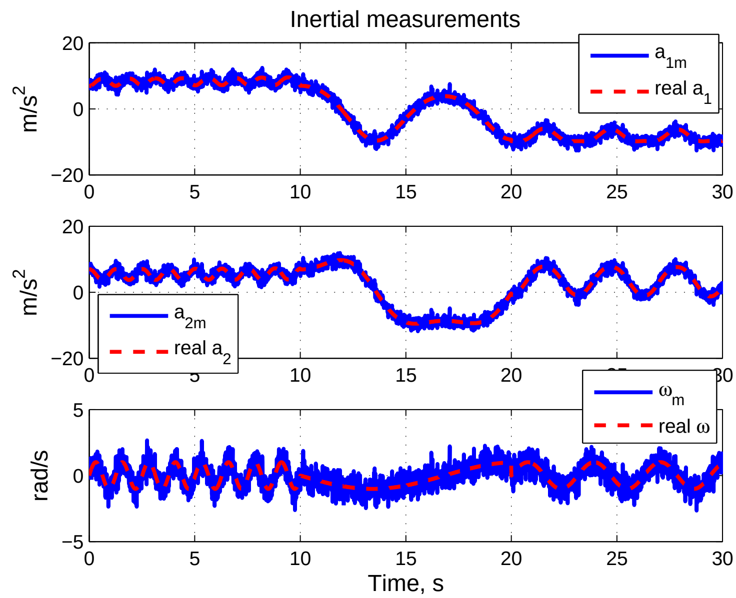

Figure 2. *Inertial measurements corrupted by a great deal of noise (simulation)*

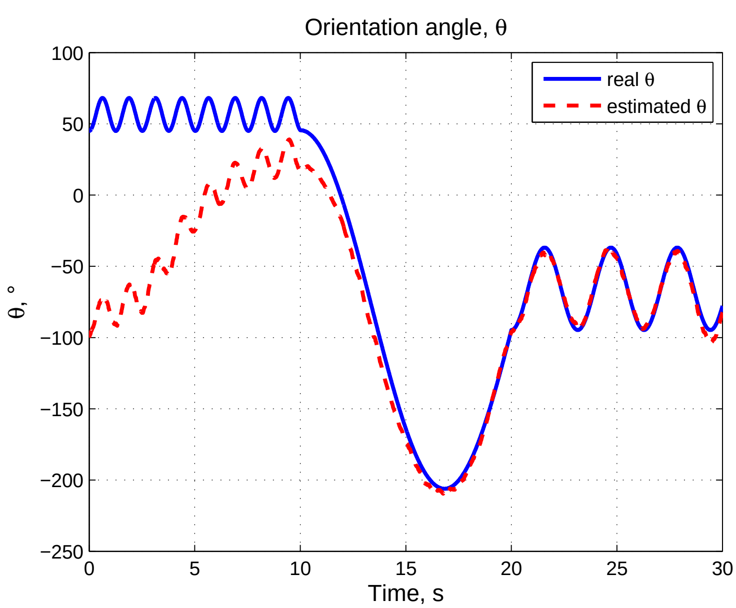

Figure 3. *Estimated and true orientation angle, θ (simulation)*

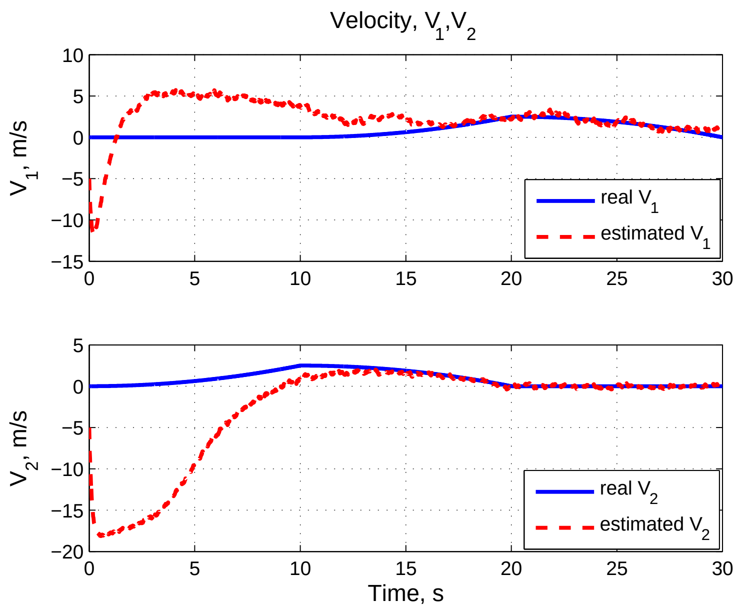

Figure 4. *Estimated and true velocity,* V_1, V_2 *(simulation)*

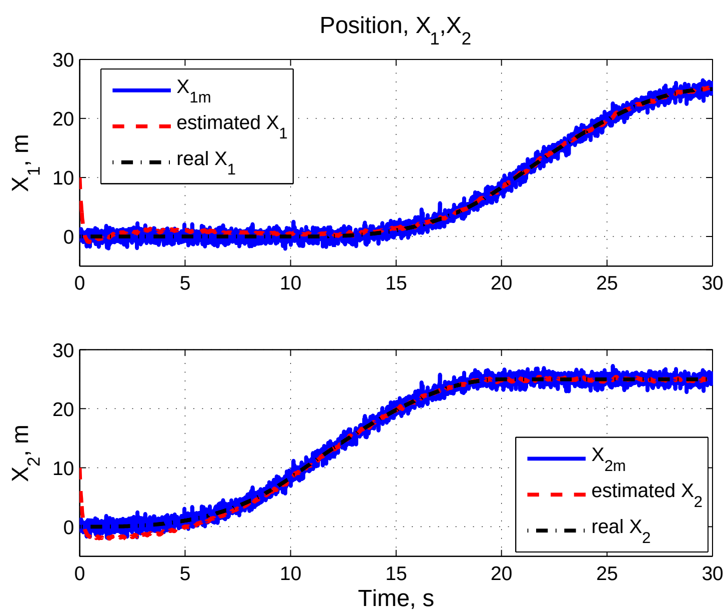

Figure 5. *Estimated, true, measured position,* X_1, X_2 *(simulation)*

inertial measurements from the IMU at a 85Hz refresh rate (*i.e.* every 12ms), whereas the update rate of position measurement is only 0.67Hz (*i.e.* every 1.5s). Between two position measurement updates, the position of the system is assumed constant, as can be seen in Figures 10,11. Even if this assumption is definitely false, it is sufficient to highlight the benefits of the proposed filter. Using a true position sensor, this update rate would be much lower (*e.g.* typically 4Hz from GPS), which would considerably improve the accuracy of the estimations. On Matlab Simulink, we feed offline the observer with the raw data and run its equations [18] at 85Hz. We have chosen the same gains as in the preceding section.

Figure 6. *Experimental setup : inertial measurement unit on a wall*

In Figures 7–11, we plot the estimations given by an observer with linear correction terms designed only around an equilibrium point, *i.e.* θ small, ("Linear" legend in the plots), and the estimations given by the symmetry-preserving observer ("Symmetry-preserving" legend in the plots). At the beginning of the experiment (t<10s), the system is left at rest with a small orientation angle, and we clearly see that the convergence behavior of both observers is very similar. However, when the system follows a highly nonlinear trajectory (value of θ becomes very large), the symmetry-preserving observer keeps providing realistic estimations, whereas the linear observer quickly diverges. Especially, the orientation angle estimated by the symmetry-preserving observer remains constant and equal to its true value as soon as the system is left at rest at t=28s, meaning that the orientation estimation error has converged to zero.

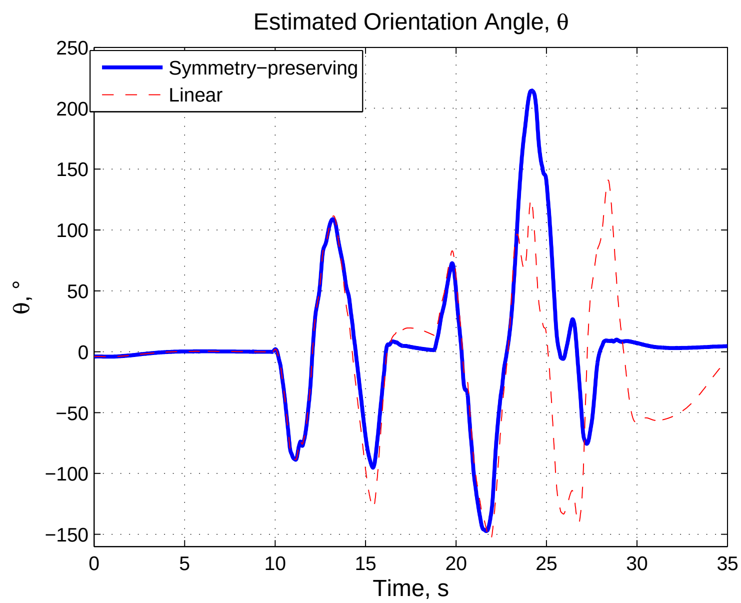

Figure 7. *Orientation angle estimation (experiment)*

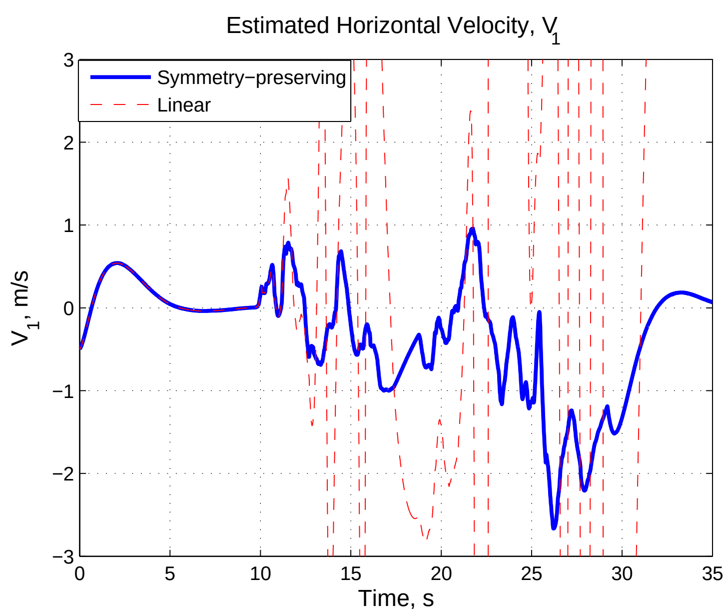

Figure 8. *Horizontal velocity V_1 (experiment)*

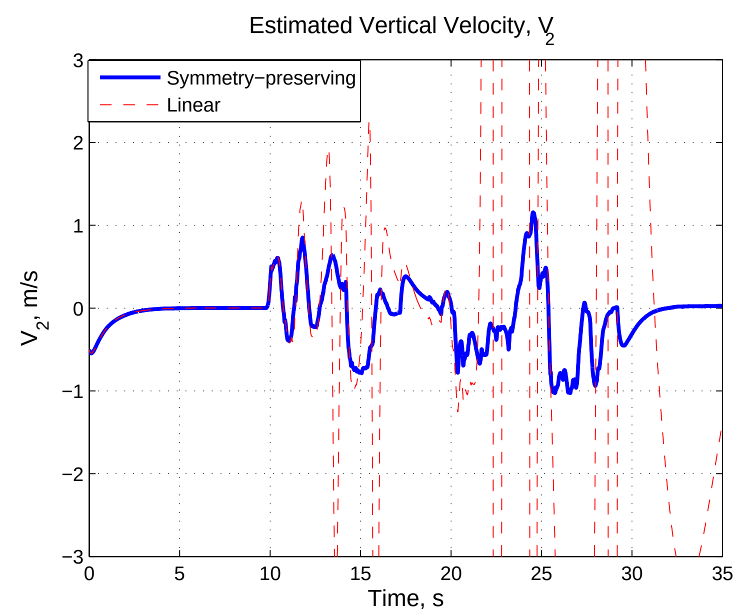

Figure 9. *Vertical velocity V_2 (experiment)*

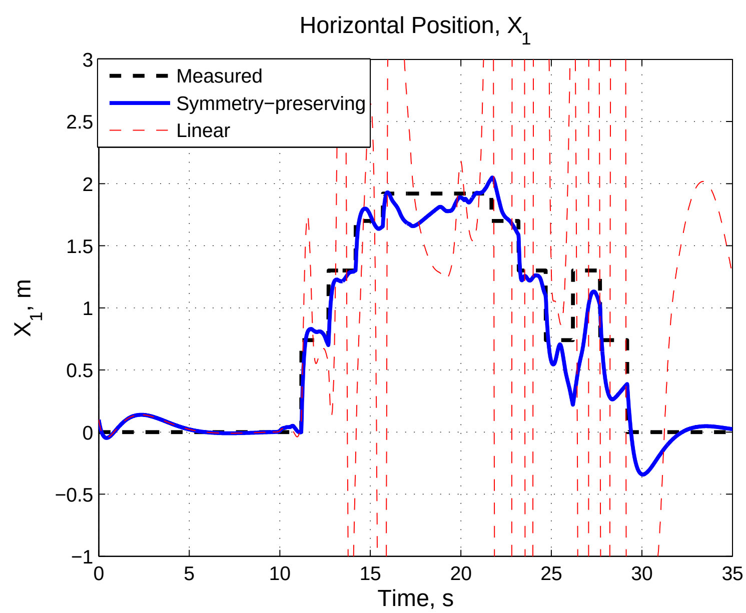

Figure 10. *Horizontal position X_1 (experiment)*

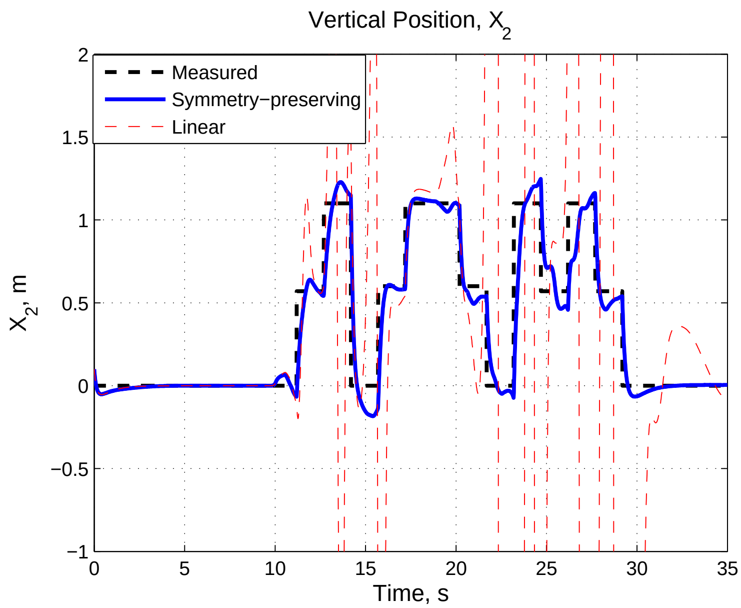

Figure 11. *Vertical position X_2 (experiment)*

6. Conclusion

In this article, we have proposed an original approach to design nonlinear observers for nonlinear dynamical systems with symmetries. We have shown how these observers can be easily derived from the design around a single equilibrium point of the considered system. Naturally preserving the symmetries of the dynamical system, such nonlinear observers behave well around any equilibrium point. This important property has been illustrated through simulation and experiments.

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