Contraction and observer design on cones

S. Bonnabel (Mines ParisTech)
Joint work with
A. Astolfi (Imperial College, University of Roma)
R. Sepulchre (University of Liège)

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Introduction

- **K is a cone in a Banach space.** Consider a system defined by a positive operator

$$x_{k+1} = f_k(x_k) \in K, \quad y_k = h_k(x_k)$$

- **Goal**: estimate the state x_k from measurements y_k s.t. the estimate $\hat{x}_k \in K$ (interpretability).
  - Motivating example: positive observers in $\mathbb{R}^n_+$ (state variables represent densities or populations, a negative estimated concentration has no physical meaning...)

- **Birkhoff theorem** characterizes a large class of contractive positive systems.

- **Idea**: a mere copy of the dynamics $\hat{x}_{k+1} = f_k(\hat{x}_k)$ may converge...

A simple illustration

Positive linear system with scalar output

$$x_{k+1} = Ax_k$$
$$y_k = c^T x_k$$

with $A_{ij} > 0 \ \forall i, j$, and $c_i > 0 \ \forall i$.

Perron-Frobenius theory (1907) :

$$\lim_{k\to\infty} x_k/\|x_k\| = u$$

where u is an eigenvector with positive coordinates.

Main idea : consider the observer

$$\hat{z}_{k+1} = A\hat{z}_k/\|A\hat{z}_k\|$$
$$\hat{x}_k = \hat{r}_k \hat{z}_k, \quad \text{with} \quad \hat{r}_k = \frac{y_k}{c^T \hat{z}_k} \simeq \|x_k\| \frac{c^T u}{c^T u}$$

A simple idea

Convergence: the rays $\frac{x_k}{\|x_k\|}, \frac{\hat{x}_k}{\|\hat{x}_k\|}$ converge towards *u*.

Moreover one can expect $\frac{\|x_k\|}{\|\hat{x}_k\|} \to 1$.

Generalization via Birkhoff theorem to

- linear time-varying systems
- a class of nonnegative matrices A
- addition of an input term Bu_k
- *any* cone K in a Banach space.

Outline

1. Birkhoff-Bushell theorem
2. Projective observer in a cone
3. Application to the positive orthant

Hilbert metric

A solid cone K defined on a Banach space $\mathcal{X}$ is a subset s.t.

- $\mathring{K} \neq \emptyset$,
- $K + K \subset K$,
- $\lambda K \subset K$ for all $\lambda \geq 0$,
- $\{-K\} \cap K = \{0\}$

Partial order: $x \leq y$ iff $y - x \in K$.

Hilbert projective metric in $K \backslash \{0\}$ is defined by

$$d(x, y) = \log[\inf\{\frac{\beta}{\alpha} \mid \alpha y \leq x \leq \beta y,\ \alpha, \beta > 0\}].$$

It defines a metric on the set of rays as $d(\lambda x, \mu y) = d(x, y)$ $\forall \lambda, \mu > 0$.

Birkhoff-Bushell theorem

Definitions Let $A : K \to K$ be a mapping:

- A is said positive if $A : \mathring{K} \to \mathring{K}$.
- A is monotone increasing if $x \leq y$ implies $A(x) \leq A(y)$.
- A is homogeneous of degree p if for all $\lambda > 0$ and $x \in K$

$$A(\lambda x) = \lambda^p A(x)$$

- The projective diameter $\Delta(A)$ is the diameter of $A(K \backslash \{0\})$.

Birkhoff-Bushell theorem

Birkhoff theorem (1957): Let A be monotone increasing and homogeneous of degree p in $\mathring{K}$. Then for all $x, y \in K$

$$d(A(x), A(y)) \leq p\, d(x, y)$$

If A is a positive linear mapping we have for all $x, y \in \mathring{K}$

$$d(Ax, Ay) \leq \gamma d(x, y), \qquad \gamma := \tanh(\frac{\Delta(A)}{4})$$

Consequences:

- if $p < 1$, A is a strict contraction.
- if $\Delta(A) < \infty$, A is a strict contraction.

Projective observer in a cone

Considered problem

Time-varying system on a solid cone K in a banach space

$$\begin{aligned} x_{k+1} &= A_k(x_k) \\ y_k &= C_k(x_k) \end{aligned}$$

where A_k is a positive homogeneous map on K and C_k is positive homogeneous of degree q.

Positive system: $x_0 \in K \Rightarrow x_k \in K, \ \forall k > 0$.

Positive observer: we want $\hat{x}_k$ to remain in K.

We start from the decomposition (polar coordinates):

$$x = rz, \quad (r, z) \in \mathbb{R}_+^* \times (\mathbb{S} \cap \mathring{K})$$

i.e. $\|z\| = 1$ and $r = \|x\| > 0$. Let $x_k = r_k z_k$. We have

$$z_{k+1} = \frac{A_k(z_k)}{\|A_k(z_k)\|}$$
$$\|y_k\| = r_k^q \|C(z_k)\|$$

Simple candidate positive observer:

$$\hat{z}_{k+1} = \frac{A_k \hat{z}_k}{\|A_k \hat{z}_k\|},$$
$$\hat{r}_k = \left(\frac{\|y_k\|}{\|C(\hat{z}_k)\|}\right)^{1/q}$$

it delivers positive estimates $\hat{x}_k = \hat{r}_k \hat{z}_k$, as $\hat{z}$ remains in $\mathring{K}$.

Convergence issues

Contraction is a powerful tool for nonlinear observer design. If the system naturally contracts, a mere copy is an observer[1].

Proposition Suppose there exists a finite horizon T such that

1. Either $\exists p\ (0 < p < 1)\ \ \forall k \in \mathbb{N}$ s.t. $A_{k+T} \circ \cdots \circ A_{k+1} \circ A_k$ is homogeneous of degree at most p.
2. Or $\exists R > 0\ \ \forall k \in \mathbb{N}\ A_{k+T} \circ \cdots \circ A_{k+1} \circ A_k$ is linear with projective diameter $\Delta \leq R$.

Then $d(\hat{z}_k, z_k) \to 0$ exponentially.

Proof: after T steps $d(\hat{z}_{T+k+1}, z_{T+k+1}) \leq \gamma d(\hat{z}_k, z_k)$ with $\gamma < 1$.

[1] see the work of Lohmiller and Slotine on contraction analysis (1999).

Application to the positive orthant

Positive linear systems

Positive orthant: $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid \forall i \; x_i \geq 0\}$.

Hilbert metric defined by

$$d(x, y) = \max_i \log(\frac{x_i}{y_i}) - \min_i \log(\frac{x_i}{y_i})$$

Linear positive system in $\mathbb{R}^n_+$

$$x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k$$

where A_k is positive, B_k and C_k are non-negative matrices.

Positive candidate observer

$$\hat{z}_0 \in \mathring{K}, \; \hat{z}_{k+1} = \frac{A_k \hat{z}_k + B_k u_k}{\|A_k \hat{z}_k + B_k u_k\|},$$

$$\hat{r}_k = \frac{\|y_k\|}{\|C_k(\hat{z}_k)\|}$$

Convergence result

Proposition: suppose there exists T such that $\forall k \in \mathbb{N}$ $A_{k+T} \circ \cdots \circ A_{k+1} \circ A_k$ is linear with projective diameter $\Delta \leq R < \infty$, then

$$d(\hat{z}_k, z_k) \to 0 \quad \text{exponentially}$$

Moreover, if $\hat{z}_k$ remains in a set with finite diameter, and $\alpha \leq \|C_k\| \leq \beta$ with $\alpha, \beta > 0$, then

$$\log \hat{r}_k - \log r_k \to 0 \quad \text{exponentially}$$

Sketch of the proof: $|\frac{r_k}{\hat{r}_k} - 1| \leq \frac{1}{\alpha \cos\theta} \mid \|C_k \hat{z}_k\| - \|C_k z_k\| \mid \leq \frac{1}{\alpha \cos\theta} \|C_k(\hat{z}_k - z_k)\| \leq \rho d(\hat{z}_k, z_k)$.

Further results: time-invariant case

Time-invariant case

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$$

Proposition : suppose that A is primitive, i.e., $\exists p \in \mathbb{N}$ $(A^p)_{ij} > 0\ \forall i, j$. Then

- $d(\hat{z}_k, z_k)$ and $|\frac{\hat{r}_k}{r_k} - 1|$ converge exponentially to zero.
- $d_{\mathring{K}}(\hat{x}_k, x_k) \to 0$ where $d_{\mathring{K}}$ is the metric on $\mathring{K}$

$$d_{\mathring{K}}(\hat{x}_k, x_k) = \sqrt{d(\hat{z}_k, z_k)^2 + |\log(\hat{r}_k/r_k)|^2}$$

Example

We consider the positive continuous-time system of (Dautrebande and Bastin, ECC99)

$$\dot{x} = \begin{pmatrix} 1 & 3 & 2 \\ 10 & 2 & 4 \\ 3 & 2 & 1 \end{pmatrix} x, \quad y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x$$

The authors prove it is not possible to build a convergent positive *linear* observer for this system.

But $\exp(A\tau) \simeq I + \tau A$ is obviously primitive, thus our *nonlinear* observer is positive and converges.

Numerical experiments

Continuous-time system of (Back and Astolfi, 2008)

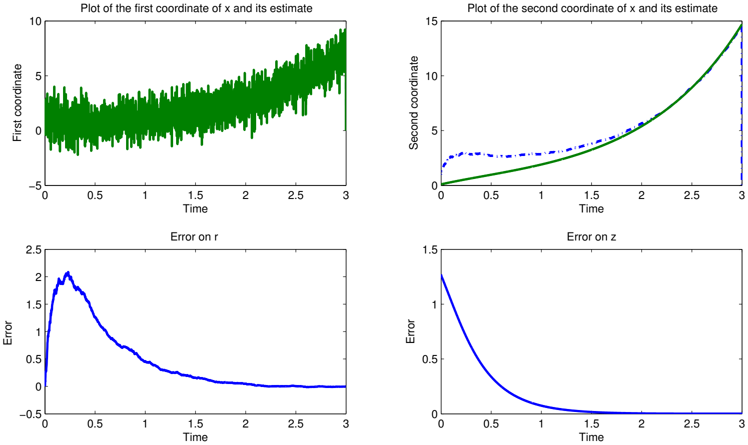
$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} x, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Convergence there exists no convergent linear positive Luenberger observer.

But $\exp(A\tau)$ is primitive for small $\tau > 0$.

Measurement noise: the estimation $\hat{z}_k$ is never noisy. The measurement noise on r_k can be easily filtered.

Numerical experiments



Top left: Measurement $y(t)$ (plain) and $\hat{x}_1(t)$ (dashed). Top right: $x_2(t)$ (plain) and $\hat{x}_2(t)$ (dashed). Bottom left: $\hat{r}/r - 1$. Bottom right: $\|\hat{z} - z\|$.

Conclusion

Conclusion

- A large class of positive maps are naturally contractive
- A cheap way to design positive asymptotic observers
- Price to pay = the convergence rate is not controlled
- Hilbert metric on the cone of hermitian positive definite matrices

$$d(X, Y) = \log\left(\frac{\lambda_{max}(XY^{-1})}{\lambda_{min}(XY^{-1})}\right)$$

- Also applied in the paper to prove convergence of quantum filters
- Other applications ?