A Separation Principle on Lie Groups

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Introduction

- Separation principle for linear systems: “an optimal stable observer, feeding an optimal stable controller $\Rightarrow$ an optimal stable feedback controller".
- Thus for a *linear time-invariant* system, combining a stable observer and a stable controller yields a stable closed-loop system.
- Does not hold for non-linear systems[1]. Holds locally around steady-states.
- We consider invariant systems on Lie groups, and we state a local separation principle around a large class of trajectories that are not necessarily steady-states.
- Control on Lie groups is well known but observer design on Lie groups have only been introduced recently.

[1] except for some classes: see Atassi and Khalil (1999),Gauthier and Kupka (1992),maithripala et al (2005)

Outline

The linear case

Consider the system

$$\frac{d}{dt}x = Ax + Bu \tag{1}$$

$$y = Cx + Du, \tag{2}$$

where $(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$. We would like to track the reference trajectory

$$\frac{d}{dt}x_r = Ax_r + Bu_r \tag{3}$$

$$y_r = Cx_r + Du_r \tag{4}$$

using only the measured output y. We want to stabilize the equilibrium $(x - x_r, u - u_r) := (0, 0)$.

The linear case

One can use the linear controller-observer

$$u = u_r - K(\hat{x} - x_r) \tag{5}$$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu - L(C\hat{x} + Du - y)$$

where the $m \times n$ matrix K and $n \times p$ matrix L are to be chosen. Setting $e_x := \hat{x} - x$ and $e_r = x - x_r$ we have

$$\frac{d}{dt}e_r = Ae_r + B(u - u_r) = Ae_r - BK(\hat{x} - x_r) = Ae_r - BK(e_x + e_r)$$

$$\frac{d}{dt}e_x = A\hat{x} + Bu - L(C\hat{x} - Cx) - (Ax + Bu) = (A - LC)e_x$$

Hence the closed-loop system

$$\frac{d}{dt}e_r = (A - BK)e_r - BKe_x \tag{6}$$

$$\frac{d}{dt}e_x = (A - LC)e_x \tag{7}$$

The last equation being autonomous, the system has a triangular structure hence its eigenvalues are those of $A - BK$ together with those of $A - LC$.

The linear case

Conclusion: If[2]

- L is such that the estimation error $e_x = \hat{x} - x$ converges exp (recall $\frac{d}{dt}e_x = (A - LC)e_x$)
- The control law $u = u_r - K(x - x_r)$ is such that the tracking error $e_r = x_r - x$ converges exp (recall in this case $\frac{d}{dt}e_r = (A - BK)e_r$)

Then

$$u = u_r - K(\hat{x} - x_r)$$

allows the system to converge to the reference trajectory (x_r, u_r, y_r) using only the measured output y.

[2]provided, (A,C) is observable and (A,B) controllable

The (non) linear case

Consider the non-linear system

$$\frac{d}{dt}x = f(x, u) \tag{8}$$

$$y = h(x, u), \tag{9}$$

We would like to track the reference trajectory

$$\frac{d}{dt}x_r = f(x_r, u_r) \tag{10}$$

$$y_r = h(x_r, u_r) \tag{11}$$

using only the measured output y. We want to stabilize the equilibrium point $(\bar{\eta}_x, \bar{\eta}_u) := (0, 0)$ of the tracking error system

$$\frac{d}{dt}\eta_x = f(x_r + \eta_x, u_r + \eta_u) - f(x_r, u_r) \tag{12}$$

$$\eta_y = h(x_r + \eta_x, u_r + \eta_u) - h(x_r, u_r), \tag{13}$$

where $\eta_x := x - x_r$, $\eta_u := u - u_r$ and $\eta_y := y - y_r$.

The (non) linear case

Linearize the error system with $\xi_x = \delta\eta_x, \xi_u = \delta\eta_u, \xi_y = \delta\eta_y$

$$\dot{\xi}_x = \partial_1 f(x_r, u_r)\xi_x + \partial_2 f(x_r, u_r)\xi_u \qquad = A\xi_x + B\xi_u \tag{14}$$

$$\xi_y = \partial_1 h(x_r, u_r)\xi_x + \partial_2 h(x_r, u_r)\xi_u \qquad = C\xi_x + D\xi_u. \tag{15}$$

And consider the observer-controller (L and K may depend on $\hat{x}$)

$$\xi_u = -K\hat{\xi}_x \tag{16}$$

$$\frac{d}{dt}\hat{\xi}_x = A\hat{\xi}_x + B\xi_u - L(C\hat{\xi}_x + D\xi_u - \xi_y)$$

Setting $e_x := \hat{\xi}_x - \xi_x$ the closed-loop system

$$\frac{d}{dt}\xi_x = (A - BK)\xi_x - BKe_x \tag{17}$$

$$\frac{d}{dt}e_x = (A - LC)e_x \tag{18}$$

has a triangular structure hence its eigenvalues are those of $A - BK$ together with those of $A - LC$.

BUT A, B, C, D are NOT time-invariant unless the reference trajectory (x_r, u_r) is an equilibrium point, i.e. $f(x_r, u_r) = 0$.

The (non) linear case

Conclusion: If

- L is such that the estimation error $e_x = \hat{x} - x$ converges exp
- The control law $u = u_r - K(x - x_r)$ is such that the tracking error $e_r = x_r - x$ converges exp

Then the control law

$$u = u_r - K(\hat{x} - x_r) \tag{19}$$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - \hat{y}), \tag{20}$$

locally stabilizes $(\bar{\eta}_x, \bar{\eta}_u) := (0, 0)$ if the reference trajectory is an equilibrium point.

The Lie group case

- A tutorial example on $SO(3)$
- General theory

A tutorial example

Consider a fully actuated rigid body in space on $SO(3)$

$$\frac{d}{dt}R = R(u \wedge \cdot) \tag{21}$$
$$y = Rb, \tag{22}$$

We would like to track the reference trajectory

$$\frac{d}{dt}R_r = R_r(u_r \wedge \cdot) \tag{23}$$
$$y_r = R_r b \tag{24}$$

using only the measured output y. We want to stabilize the equilibrium point $(\bar{\eta}_R, \bar{\eta}_u) := (Id, 0)$ of the tracking error system

$$\frac{d}{dt}\eta_R = -(u_r \wedge \cdot)\eta_R + \eta_R(u \wedge \cdot) \tag{25}$$
$$\eta_y = R^{-1} y_r - b, \tag{26}$$

where $\eta_R := R_r^T R$, $\eta_u := u - u_r$ and $\eta_y := \eta_R b - b$.

A tutorial example

Consider the observer[3] on the Lie group $SO(3)$

$$\frac{d}{dt}\hat{R} = \hat{R}\big(u + L(\hat{R}^{-1}y)\big) \wedge \cdot$$

Letting $\epsilon_R = R^{-1}\hat{R}$ we have the estimation error equation

$$\frac{d}{dt}\epsilon_R = -(u \wedge \cdot)R^{-1}\hat{R} + R^{-1}\frac{d}{dt}\hat{R} = -(u \wedge \cdot)\epsilon_R + \epsilon_R((u + L\epsilon_R^{-1}b) \wedge \cdot)$$

as $y = Rb$. Letting $\epsilon_R \simeq Id + e_R \wedge \cdot$ we have up to second order terms in e_R, $u - u_r$

$$\frac{d}{dt}e_R = -u_r \wedge e_R - L(e_R \wedge b)$$

which is autonomous if u_r is constant (reminds of $\frac{d}{dt}e_x = (A - LC)e_x$).

[3] see e.g. Mahony, Hamel, Pflimlin (CDC 2005, IEEE-TAC 2008), Vasconcelos, Silvestre and Oliveira (CDC 2008), Martin, Salaun (CDC 2008), Bonnabel, Martin, Rouchon (IEEE-TAC 2008)

A tutorial example

Linearize the tracking error system. Let $R_r^{-1}R \simeq Id + (\xi_R \wedge \cdot)$ and $\xi_u = u - u_r$

$$\dot{\xi}_R = -u_r \wedge \xi_R + u - u_r = A(u_r)\xi_R + B\xi_u \quad (27)$$

$$\xi_y = \xi_R \wedge b = C\xi_R. \quad (28)$$

Considering in addition the linear controller where $R_r^{-1}\hat{R} \simeq Id + \hat{\xi}_R \wedge \cdot$

$$\xi_u = -K\hat{\xi}_R = -K(\xi_R + e_R)$$

as $R_r^{-1}\hat{R} = (R_r^{-1}R)(R^{-1}\hat{R})$. The closed-loop system is

$$\dot{\xi}_R = -u_r \wedge \xi_R - K(\xi_R + e_R) = (A - BK)\xi_R - BKe_R \quad (29)$$

$$\dot{e}_R = -u_r \wedge e_R - L(e_R \wedge b) = (A - LC)e_R \quad (30)$$

has a triangular structure. Moreover A, B, C, D are time-invariant as soon as u_r is constant.

A tutorial example

Conclusion: If[4] the observer and the controller locally converge around (R_r, u_r) control law

$$u = u_r - K\kappa(R^{-1}\hat{R})$$
$$\frac{d}{dt}\hat{R} = \hat{R}(u \wedge \cdot) + \hat{R}(L(\hat{R}^{-1}y) \wedge \cdot)$$

then locally stabilizes (R, u) around the reference trajectory (R_r, u_r) generated by

$$u_r \equiv \text{const}$$

using only the measured output y.

NB: Here κ is the logarithm map of the group (i.e. the inverse of the exponential map of the group).

[4] provided, (A,C) is observable and (A,B) controllable

The Lie group case: theory

- In the general non-linear case the matrices A, B, C, D of the linearized system depend on (x_r, u_r).
- In the case of the fully-actuated system on $SO(3)$ the matrices A, B, C, D depend only on (u_r)
- Is it logical ??

A similar result can be proved in the general case.

The Lie group case: theory

We consider the system where the state is a Lie group G

$$\frac{d}{dt}x = f(x,u)$$
$$y = h(x,u),$$

Definition

Let Σ be an open set (or more generally a manifold) and G a Lie group. A transformation group on Σ is a family of smooth maps

$$\xi \in \Sigma \mapsto \phi_x(\xi) \in \Sigma$$

such that

- $\phi_e(\xi) = \xi$ *for all* ξ
- $\phi_{x_2} \circ \phi_{x_1}(\xi) = \phi_{x_2 x_1}(\xi)$ *for all* $x_1, x_2 \in G$ *and* $\xi \in \Sigma$.

The Lie group case: theory

Consider the transformation group on $G \times \mathcal{U} \times \mathcal{Y}$ defined by

$$\phi_{x_0}(x, u, y) := \big(x_0 x, \psi_{x_0}(u), \varrho_{x_0}(y)\big)$$

We will assume the system is invariant to this group action i.e. for all x_0, x, u letting

$$(X, U, Y) := \big(x_0 x, \psi_{x_0}(u), \varrho_{x_0}(y)\big)$$

we have

$$\begin{aligned} \frac{d}{dt} X &= f(X, U) \\ Y &= h(X, U), \end{aligned}$$

the system is unchanged by the transformation.

The Lie group case: theory

For those invariant systems on Lie group one can build invariant observers[5].

- Those symmetry-preserving observers are such that the error system around the so-called permanent trajectories defined by

$$I_r = \psi_{x_r^{-1}}(u_r) \equiv const$$

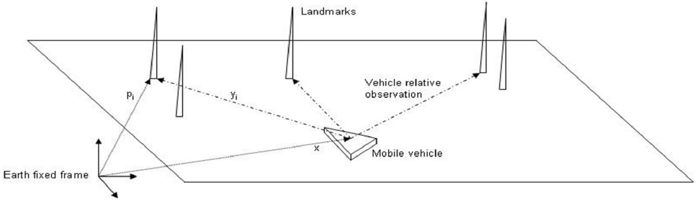
  is time-invariant[6].
- Fortunately, the tracking error system is also time-invariant around permanent trajectories.
- Thus a separation principle holds around permanent trajectories.
- Note that, in the previous example we had $I_r = u_r$.

[5] see e.g. Mahony, Hamel, Pflimlin (CDC 2005, IEEE-TAC 2008), Vasconcelos, Silvestre and Oliveira (CDC 2008), Martin, Salaun (CDC 2008), Bonnabel, Martin, Rouchon (IEEE-TAC 2008)

[6] in Bonnabel, Martin, Rouchon: Symmetry-preserving observers on Lie groups, IEEE-TAC (2009)

Examples

Localizing from landmarks using sonar



- We consider a non-holonomic car localizing from measurements of relative position to known landmarks.
- The group is $SE(2)$.
- A controller and an invariant observer were designed.
- The permanent trajectories are lines and circles with constant speed.
- Around those primitives a separation principle holds and the observer-controller is proved to be stable.

Simple mechanical systems on Lie groups

We consider fully-actuated systems described by the so-called Euler-Poincaré equations[7]

$$\frac{d}{dt}x = f(x, \xi) \tag{31}$$

$$\frac{d}{dt}\xi = A(\xi) + I^{-1}(F(x, \xi) + u) \tag{32}$$

where

- $x \in G$ is the (generalized) position, with G a Lie group (the configuration space),
- $\xi \in T_x G$ is the (generalized) velocity,
- $I^{-1}(F(x, \xi) + u)$ is the resultant force acting on the system,
- $u \in T_x G$ denotes the control.

It does not directly fit in the framework, but the results were can be extended in a special case.

[7] as discribed by e.g. Bullo, Murray (1999)

Conclusion

- We proved a local separation principle around a large set of trajectories for non-linear invariant systems on Lie groups.
- A link was established between observers on Lie groups and control on Lie groups.
- In future research we plan to explore examples of mechanical systems for which those results apply.