# Invariant Extended Kalman Filter: Theory and application to a velocity-aided estimation problem

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# Introduction

Symmetries were used in control for feedback design but much less for observer design.

When a system possesses symmetries, the standard extended Kalman filter generally does not preserve the symmetries.

For a non-linear system possessing symmetries, additive white noise does not preserve the symmetries.

## The extended Kalman filter (EKF)

The system is defined by a stochastic differential equation,

$$\dot{x} = f(x,u) + M(x)w$$
$$y = h(x,u) + N(x)v,$$

where $x, u, y$ belong to an open subset of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$; $w, v$ are white gaussian noises. Standard EKF equations:

$$\dot{\hat{x}} = f(\hat{x},u) + K \cdot \big(y - h(\hat{x},u)\big) = F(\hat{x},u,y)$$

$K$ ? Compute the gain $K$ as in a linear Kalman filter since the estimation error $\Delta x = \hat{x} - x$ satisfies up to higher order terms the linear equation

$$\Delta\dot{x} = (A - KC)\Delta x - Mw + KNv. \qquad (1)$$

$$A = \partial_1 f(\hat{x},u), \quad C = \partial_1 h(\hat{x},u), K = PC^T(NN^T)^{-1}$$
$$\dot{P} = AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP,$$

What about this “linear" approach when the state space is a manifold, a group ??

## An example: GPS-aided inertial navigation

The motion of a rigid body is

$$\frac{d}{dt}R = R(\omega \wedge \cdot)$$
$$\frac{d}{dt}V = A + Ra$$

where

- $R \in SO(3)$ is the orientation of the body mapping the body frame to earth frame
- $V$ is the velocity with respect to earth frame
- $\omega$ is the angular velocity measured by gyros
- $a$ is the specific acceleration measured by acceleros
- $A = (0\ 0\ g)^T$ is the constant gravity vector in North-East-Down (NED) coordinates

Quaternions are well suited to calculations and computer implementation

# Use of the quaternions

$$p = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix} \qquad \mathbb{R}^3 \ni \vec{p} \equiv \begin{pmatrix} 0 \\ \vec{p} \end{pmatrix} \in \mathbb{H}$$

Multiplication law :

$$p * q := \begin{pmatrix} p_0 q_0 - \vec{p} \cdot \vec{q} \\ p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q} \end{pmatrix}.$$

Unit element :

$$e := \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix},$$

To any quaternion $q$ whose norm is 1 , we can associate a certain rotation matrix $R_q \in SO(3)$ thanks to the following formula

$$q^{-1} * \vec{p} * q = R_q \cdot \vec{p} \quad \text{pour tout } \vec{p}.$$

# The considered system: GPS/IMU fusion

To design our observers we consider the system

$$\begin{aligned}\dot{q} &= q * (\omega_m - \omega_b)\\ \dot{V} &= A + \frac{1}{a_s} q * a_m * q^{-1}\\ \dot{\omega}_b &= 0\\ \dot{a}_s &= 0,\end{aligned}$$

where $\omega_m$ and $a_m$ are seen as known inputs, together with the output

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V \\ q^{-1} * B * q \end{pmatrix}.$$

where $B$ is the constant earth magnetic field, measured by magnetometers.

# The multiplicative extended Kalman filter (MEKF)

- The linear error $\Delta q = \hat{q} - q$ does not have much sense for quaternion.
- The EKF update does not preserve $\|\hat{q}\| = 1$.

Well-known MEKF[1] based on the group error $q^{-1} * \hat{q}$

$$\frac{d}{dt}\hat{q} = \hat{q} * (\omega_m - \hat{\omega}_b) + \hat{q} * K_q E$$

$$\frac{d}{dt}\hat{V} = A + \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1} + K_V E$$

$$\frac{d}{dt}\hat{\omega}_b = K_\omega E, \qquad \frac{d}{dt}\hat{a}_s = K_a E.$$

with

$$E = \begin{pmatrix} \hat{y}_V - y_V \\ \hat{y}_B - y_B \end{pmatrix}.$$

[1] E. Lefferts, F. Markley, and M. Shuster, Kalman filtering for spacecraft attitude, 1982.
Y. Huang, F. Chang, and L. Wang, The attitude determination algorithm using integrated GPS/INS data, IFAC 2005.

# The multiplicative extended Kalman filter (MEKF)

Let us suppose the noise enters the system as

$$\begin{aligned}
\frac{d}{dt}q &= q * (\omega_m - \omega_b) + q * M_q w_q \\
\frac{d}{dt}V &= A + \frac{1}{a_s} q * a_m * q^{-1} + q * M_V w_V * q^{-1} \\
\frac{d}{dt}\omega_b &= M_\omega w_\omega \\
\frac{d}{dt}a_s &= M_a w_a,
\end{aligned}$$

and the output as

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V + N_V v_V \\ q^{-1} * B * q + N_B v_B \end{pmatrix},$$

with $M_q, M_V, M_\omega, N_V, N_B$ diagonal matrices. The driving and observation noises are thus consistent with a scalar additive noise on each individual sensor.

# The multiplicative extended Kalman filter (MEKF)

Tuning ? Matrices $A$, $C$ ?

The state error $\mu = q^{-1} * \hat{q}$, $\nu = \hat{V} - V$, $\beta = \hat{\omega}_b - \omega_b$ and $\alpha = \hat{a}_s - a_s$ yields the error system around $(\mu, \nu, \beta, \alpha) = (1, 0, 0, 0)$:

$$\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{\nu} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \end{pmatrix} = (A - KC) \begin{pmatrix} \delta\mu \\ \delta\nu \\ \delta\beta \\ \delta\alpha \end{pmatrix} - M \begin{pmatrix} w_q \\ w_V \\ w_\omega \\ w_a \end{pmatrix} + KN \begin{pmatrix} v_V \\ v_B \end{pmatrix},$$

which has the desired form with $A$, $C$, $M$, $N$ depending on $\hat{q}$, $\omega_m$, $a_m$.

# Features of the MEKF

- Sound geometric structure for the quaternion estimation equation by construction it preserves the unit norm of the estimated quaternion.
- Driving noise is a sensor noise.
- Possible convergence issues in many situations. Indeed, the matrices $A$ and $C$ used for computing the gain matrix $K$ are constant only in level flight.

# Invariant Extended Kalman filter

Provides a geometric framework to the MEKF.

We notice the state space is a group $G$ for the law given by

$$\begin{pmatrix} p_0 \\ V_0 \\ \omega_0 \\ a_0 \end{pmatrix} \diamond \begin{pmatrix} q \\ V \\ \omega_b \\ a_s \end{pmatrix} := \begin{pmatrix} p_0 * q \\ p_0 * (V + V_0) * p_0^{-1} \\ \omega_b + \omega_0 \\ a_s a_0 \end{pmatrix},$$

The physical meaning is clear: rotation and translation in Earth axes, translation in body axes, and scaling.

We also consider the group transformation

$$\psi_{(p_0, V_0, \omega_0, a_0)} \begin{pmatrix} \omega_m \\ a_m \\ A \\ B \end{pmatrix} = \begin{pmatrix} \omega_m + \omega_0 \\ a_0 a_m \\ p_0 * A * p_0^{-1} \\ p_0 * B * p_0^{-1} \end{pmatrix}$$

$$\rho_{(p_0, V_0, \omega_0, a_0)} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} p_0 * (y_V + V_0) * p_0^{-1} \\ y_B \end{pmatrix}.$$

# Invariant Extended Kalman filter

Let $g = (p, V, w_b, a_s)^T \in G$ denote the state. The system with noise turned off writes

$$\frac{d}{dt} g = f(g, u)$$
$$y = h(g, u)$$

with $u = (w_m, a_m, A, B)^T$. Let $g_0 = (p_0, V_0, w_0, a_0)^T$. The system is invariant to the transformation above. Indeed let

$$g_1 = g_0 \diamond g, \quad u_1 = \psi_{g_0}(u), \quad y_1 = \rho_{g_0}(y)$$

We have the same system (the system possesses symmetries):

$$\frac{d}{dt} g_1 = f(g_1, u_1)$$
$$y_1 = h(g_1, u_1)$$

# Invariant Extended Kalman filter

An EKF writes $\frac{d}{dt}\hat{g} = F(\hat{g}, u, y)$. For any $g_0 \in G$ let

$$\hat{g}_1 = g_0 \diamond \hat{g}, \quad u_1 = \psi_{g_0}(u), \quad y_1 = \rho_{g_0}(y)$$

We want the same formula in the new variables

$$\frac{d}{dt}\hat{g}_1 = F(\hat{g}_1, u_1, y_1)$$

Let $L_{g_1}(g) = g_1 \diamond g$ be the left multiplication on $G$. To be *invariant* to the transformation the EKF must write[2]

$$\frac{d}{dt}\hat{g} = f(\hat{g}, u) + DL_{\hat{g}}(e) \cdot K \cdot \Big(\rho_{\hat{g}^{-1}}(y) - \rho_{\hat{g}^{-1}}(h(\hat{g}, u))\Big),$$

where the matrix gain $K$ may depend only on $\hat{I} = \psi_{\hat{g}^{-1}}(u),\ E$.
The observer has the same geometric structure as the system !

[2]Bonnabel, Martin, Rouchon. Symmetry-preserving observers. IEEE-TAC. 2008.

# IEKF: How do we tune the gains ??

Define the invariant intrinsically defined error

$$\eta = g^{-1} \diamond \hat{g}$$

Linearize it for $\hat{g}$ and $g$ close ($\eta$ close to $e$).

$$\frac{d}{dt}\delta\eta = (A - KC)\delta\eta$$

with

$$C := \partial_1 h(e, \hat{I}), \quad A\xi := \left[\xi, f(e, \hat{I})\right] - \partial_1 f(e, \hat{I}) \cdot \partial_1 \psi(e, \hat{I}) \cdot \xi$$

and we set

$$\dot{P} = AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP,$$
$$K = PC^T(NN^T)^{-1}$$

Still, how do we choose $M, N$ ??

## Back to the example: IEKF structure

$$\begin{aligned}
\frac{d}{dt}\hat{q} &= \hat{q} * (\omega_m - \hat{\omega}_b) + \hat{q} * (K_q E) \\
\frac{d}{dt}\hat{V} &= A + \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1} + \hat{q} * (K_V E) * \hat{q}^{-1} \\
\frac{d}{dt}\hat{\omega}_b &= K_\omega E \\
\frac{d}{dt}\hat{a}_s &= \hat{a}_s K_a E,
\end{aligned}$$

$$E = \rho_{\hat{x}^{-1}} \begin{pmatrix} \hat{y}_V \\ \hat{y}_B \end{pmatrix} - \rho_{\hat{x}^{-1}} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} \hat{q}^{-1} * (\hat{V} - y_V) * \hat{q} \\ \hat{q}^{-1} * B * \hat{q} - y_B \end{pmatrix}.$$

The invariant state error $g^{-1} \diamond \hat{g}$ reads

$$\begin{pmatrix} \mu \\ \nu \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} q^{-1} * \hat{q} \\ q^{-1} * (\hat{V} - V) * q \\ \hat{\omega}_b - \omega_b \\ \frac{\hat{a}_s}{a_s} \end{pmatrix},$$

hence we recover the quaternion error used in the MEKF.

# Back to the example: Features of the IEKF

Symmetry-preserving structure rotations, translations and scaling in the appropriated frames leave the error system unchanged, which is meaningful from an engineering point of view.

Sound geometric structure for the quaternion estimation equation: it preserves the unit norm of the estimated quaternion.

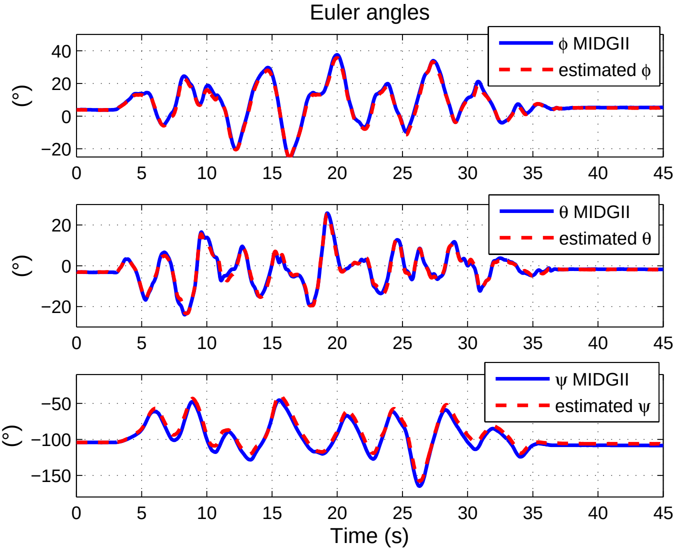
Larger expected domain of convergence

Proposition: the matrices $A$ and $C$ used for computing the gain matrix $K$ are constant not only in level flight but also on a large set of trajectories (uniform acceleration, rotation with constant angular velocity...).

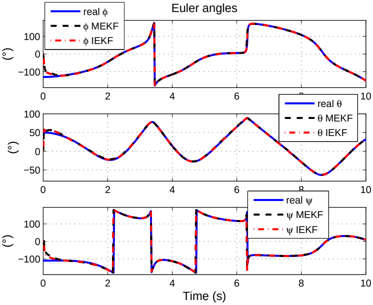
$$\text{Recall} \quad \frac{d}{dt}\delta\eta = (A - KC)\delta\eta$$

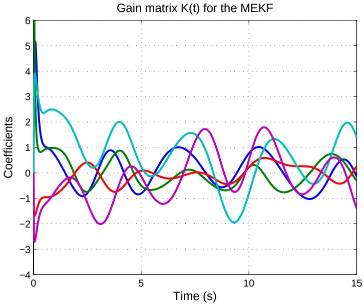
# Back to the example: Numerical results

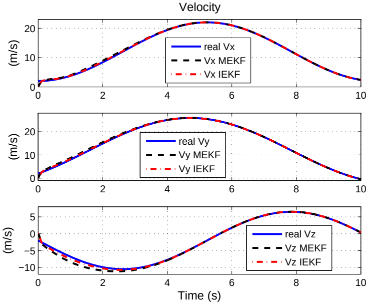
Experiment: estimated Euler angles. Comparison with a commercial INS-GPS device MIDG2.

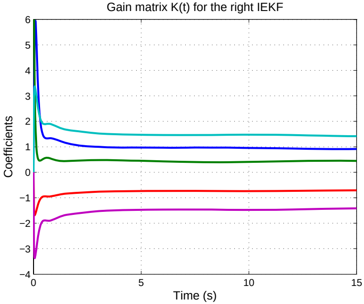


# Simulation results: comparison of MEKF and IEKF



# IEKF: what about the noise matrices $M$, $N$?

The system with noise turned off

$$\frac{d}{dt}g = f(g,u) + M(g)w$$
$$y = h(g,u) + N(y)v$$

is invariant to the transformation

$$g_1 = g_0 \diamond g, \quad u_1 = \psi_{g_0}(u), \quad y_1 = \rho_{g_0}(y)$$

It seems logical that the system with noise be invariant as well i.e.

$$\frac{d}{dt}g = f(g_1,u_1) + M(g_1)w$$
$$y = h(g_1,u_1) + N(y_1)v$$

In particular we take $\frac{d}{dt}\hat{g} = f(\hat{g},u) + DL_{\hat{g}}Mw$. On the example it yields the same driving noise as for the MEKF.

# IEKF: what about the noise matrices $M, N$?

With this definition of the noise matrices, the linearized error equation is a stochastic multiplicative linear differential equation

$$\frac{d}{dt}\delta\eta = (A - KC)\delta\eta + Q_1\big(\delta\eta, M(e)w\big) + Q_2\big(\delta\eta, KN(e)v\big),$$

It is NOT the linear model for which the KF is built

$$\frac{d}{dt}\delta\eta = (A - KC)\delta\eta - M(e)w + KN(e)v.$$

Proposition: For both equations, the mean and covariance of the $\delta\eta$ are the same up to second order terms in the noise amplitude

Thus $M, N$ can be chosen on the non-linear system, then build a Kalman filter with on the linearized system etc.

# Conclusion

- When a system possesses symmetries, compell the EKF to preserve them is a way to provide it with the rich geometric structure of the physical system.
- On a group or a manifold more logical than the usual EKF based on the linear error $\hat{x} - x$
- Corresponds to a time-invariant linear Kalman filter around a whole set of trajectories
- Particularly suited to aerospace (UAVs) applications, (Symmetries = Galilean invariances).