

# Invariant Extended Kalman Filter: Theory and application to a velocity-aided estimation problem

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# Introduction

**Symmetries** were used in control for feedback design but much less for observer design.

When a system possesses symmetries, the standard **extended Kalman filter** generally does not preserve the symmetries.

For a **non-linear system** possessing symmetries, **additive white noise** does not preserve the symmetries.

## The extended Kalman filter (EKF)

The system is defined by a stochastic differential equation,

$$\begin{aligned}\dot{x} &= f(x, u) + M(x)w \\ y &= h(x, u) + N(x)v,\end{aligned}$$

where  $x, u, y$  belong to an open subset of  $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ ;  
 $w, v$  are white gaussian noises. Standard EKF equations:

$$\dot{\hat{x}} = f(\hat{x}, u) + K \cdot (y - h(\hat{x}, u)) = F(\hat{x}, u, y)$$

$K$  ? Compute the gain  $K$  as in a **linear** Kalman filter since the estimation error  $\Delta x = \hat{x} - x$  satisfies up to higher order terms the linear equation

$$\Delta \dot{x} = (A - KC)\Delta x - Mw + KNv. \quad (1)$$

$$\begin{aligned}A &= \partial_1 f(\hat{x}, u), \quad C = \partial_1 h(\hat{x}, u), \quad K = PC^T(NN^T)^{-1} \\ \dot{P} &= AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP,\end{aligned}$$

What about this “linear” approach when the state space is a manifold, a group ??

## An example: GPS-aided inertial navigation

The motion of a rigid body is

$$\frac{d}{dt}R = R(\omega \wedge \cdot)$$

$$\frac{d}{dt}V = A + Ra$$

where

- ▶  $R \in SO(3)$  is the orientation of the body mapping the body frame to earth frame
- ▶  $V$  is the velocity with respect to earth frame
- ▶  $\omega$  is the angular velocity measured by gyros
- ▶  $a$  is the specific acceleration measured by acceleros
- ▶  $A = (0 \ 0 \ g)^T$  is the constant gravity vector in North-East-Down (NED) coordinates

Quaternions are well suited to calculations and computer implementation

# Use of the quaternions

$$p = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix} \quad \mathbb{R}^3 \ni \vec{p} \equiv \begin{pmatrix} 0 \\ \vec{p} \end{pmatrix} \in \mathbb{H}$$

Multiplication law :

$$p * q := \begin{pmatrix} p_0 q_0 - \vec{p} \cdot \vec{q} \\ p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q} \end{pmatrix}.$$

Unit element :

$$e := \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix},$$

To any quaternion  $q$  whose norm is 1 , we can associate a certain rotation matrix  $R_q \in SO(3)$  thanks to the following formula

$$q^{-1} * \vec{p} * q = R_q \cdot \vec{p} \quad \text{pour tout } \vec{p}.$$

# The considered system: GPS/IMU fusion

To design our observers we consider the system

$$\dot{q} = q * (\omega_m - \omega_b)$$

$$\dot{V} = A + \frac{1}{a_s} q * a_m * q^{-1}$$

$$\dot{\omega}_b = 0$$

$$\dot{a}_s = 0,$$

where  $\omega_m$  and  $a_m$  are seen as known inputs, together with the output

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V \\ q^{-1} * B * q \end{pmatrix}.$$

where  $B$  is the constant earth magnetic field, measured by magnetometers.

## The multiplicative extended Kalman filter (MEKF)

- ▶ The **linear** error  $\Delta q = \hat{q} - q$  does not have much sense for quaternion.
- ▶ The EKF update does not preserve  $\|\hat{q}\| = 1$ .

Well-known MEKF<sup>1</sup> based on the **group** error  $q^{-1} * \hat{q}$

$$\begin{aligned}\frac{d}{dt}\hat{q} &= \hat{q} * (\omega_m - \hat{\omega}_b) + \hat{q} * K_q E \\ \frac{d}{dt}\hat{V} &= A + \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1} + K_V E \\ \frac{d}{dt}\hat{\omega}_b &= K_\omega E, \quad \frac{d}{dt}\hat{a}_s = K_a E.\end{aligned}$$

with

$$E = \begin{pmatrix} \hat{y}_V - y_V \\ \hat{y}_B - y_B \end{pmatrix}.$$

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<sup>1</sup>E. Lefferts, F. Markley, and M. Shuster, Kalman filtering for spacecraft attitude, 1982.

Y. Huang, F. Chang, and L. Wang, The attitude determination algorithm using integrated GPS/INS data, IFAC 2005.

# The multiplicative extended Kalman filter (MEKF)

Let us suppose the noise enters the system as

$$\frac{d}{dt} \mathbf{q} = \mathbf{q} * (\omega_m - \omega_b) + \mathbf{q} * M_q \mathbf{w}_q$$

$$\frac{d}{dt} V = A + \frac{1}{a_s} \mathbf{q} * a_m * \mathbf{q}^{-1} + \mathbf{q} * M_V \mathbf{w}_V * \mathbf{q}^{-1}$$

$$\frac{d}{dt} \omega_b = M_\omega \mathbf{w}_\omega$$

$$\frac{d}{dt} a_s = M_a \mathbf{w}_a,$$

and the output as

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V + N_V v_V \\ \mathbf{q}^{-1} * B * \mathbf{q} + N_B v_B \end{pmatrix},$$

with  $M_q, M_V, M_\omega, N_V, N_B$  diagonal matrices. The driving and observation noises are thus consistent with a scalar additive noise on each individual sensor.

# The multiplicative extended Kalman filter (MEKF)

Tuning ? Matrices  $A$ ,  $C$  ?

The state error  $\mu = q^{-1} * \hat{q}$ ,  $\nu = \hat{V} - V$ ,  $\beta = \hat{\omega}_b - \omega_b$  and  $\alpha = \hat{a}_s - a_s$  yields the error system around  $(\mu, \nu, \beta, \alpha) = (1, 0, 0, 0)$ :

$$\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{\nu} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \end{pmatrix} = (A - KC) \begin{pmatrix} \delta\mu \\ \delta\nu \\ \delta\beta \\ \delta\alpha \end{pmatrix} - M \begin{pmatrix} w_q \\ w_V \\ w_\omega \\ w_a \end{pmatrix} + KN \begin{pmatrix} v_V \\ v_B \end{pmatrix},$$

which has the desired form with  $A$ ,  $C$ ,  $M$ ,  $N$  depending on  $\hat{q}$ ,  $\omega_m$ ,  $a_m$ .

# Features of the MEKF

- ▶ **Sound geometric structure for the quaternion estimation equation** by construction it preserves the unit norm of the estimated quaternion.
- ▶ **Driving noise** is a sensor noise.
- ▶ **Possible convergence issues in many situations.** Indeed, the matrices  $A$  and  $C$  used for computing the gain matrix  $K$  are constant only in level flight.

# Invariant Extended Kalman filter

Provides a **geometric framework** to the MEKF.

We notice the state space is a group  $G$  for the law given by

$$\begin{pmatrix} p_0 \\ V_0 \\ \omega_0 \\ a_0 \end{pmatrix} \diamond \begin{pmatrix} q \\ V \\ \omega_b \\ a_s \end{pmatrix} := \begin{pmatrix} p_0 * q \\ p_0 * (V + V_0) * p_0^{-1} \\ \omega_b + \omega_0 \\ a_s a_0 \end{pmatrix},$$

The physical meaning is clear: rotation and translation in Earth axes, translation in body axes, and scaling.

We also consider the group transformation

$$\psi(p_0, V_0, \omega_0, a_0) \begin{pmatrix} \omega_m \\ a_m \\ A \\ B \end{pmatrix} = \begin{pmatrix} \omega_m + \omega_0 \\ a_0 a_m \\ p_0 * A * p_0^{-1} \\ p_0 * B * p_0^{-1} \end{pmatrix}$$
$$\rho(p_0, V_0, \omega_0, a_0) \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} p_0 * (y_V + V_0) * p_0^{-1} \\ y_B \end{pmatrix}.$$

# Invariant Extended Kalman filter

Let  $g = (p, V, w_b, a_s)^T \in G$  denote the state. The system with noise turned off writes

$$\begin{aligned}\frac{d}{dt}g &= f(g, u) \\ y &= h(g, u)\end{aligned}$$

with  $u = (w_m, a_m, A, B)^T$ . Let  $g_0 = (p_0, V_0, w_0, a_0)^T$ . The system is **invariant to the transformation** above. Indeed let

$$g_1 = g_0 \diamond g, \quad u_1 = \psi_{g_0}(u), \quad y_1 = \rho_{g_0}(y)$$

We have the **same** system (the system possesses symmetries):

$$\begin{aligned}\frac{d}{dt}g_1 &= f(g_1, u_1) \\ y_1 &= h(g_1, u_1)\end{aligned}$$

# Invariant Extended Kalman filter

An EKF writes  $\frac{d}{dt}\hat{g} = F(\hat{g}, u, y)$ . For any  $g_0 \in G$  let

$$\hat{g}_1 = g_0 \diamond \hat{g}, \quad u_1 = \psi_{g_0}(u), \quad y_1 = \rho_{g_0}(y)$$

We want the **same** formula in the new variables

$$\frac{d}{dt}\hat{g}_1 = F(\hat{g}_1, u_1, y_1)$$

Let  $L_{g_1}(g) = g_1 \diamond g$  be the left multiplication on  $G$ . To be **invariant** to the transformation the EKF must write<sup>2</sup>

$$\frac{d}{dt}\hat{g} = f(\hat{g}, u) + DL_{\hat{g}}(e) \cdot K \cdot \left( \rho_{\hat{g}^{-1}}(y) - \rho_{\hat{g}^{-1}}(h(\hat{g}, u)) \right),$$

where the matrix gain  $K$  may depend only on  $\hat{g} = \psi_{\hat{g}^{-1}}(u)$ ,  $E$ .  
The observer has **the same geometric structure** as the system !

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<sup>2</sup>Bonnabel, Martin, Rouchon. Symmetry-preserving observers. IEEE-TAC. 2008.

## IEKF: How do we tune the gains ??

Define the **invariant intrinsically defined** error

$$\eta = g^{-1} \diamond \hat{g}$$

Linearize it for  $\hat{g}$  and  $g$  close ( $\eta$  close to  $e$ ).

$$\frac{d}{dt} \delta \eta = (A - KC) \delta \eta$$

with

$$C := \partial_1 h(e, \hat{l}), \quad A\xi := [\xi, f(e, \hat{l})] - \partial_1 f(e, \hat{l}) \cdot \partial_1 \psi(e, \hat{l}) \cdot \xi$$

and we set

$$\begin{aligned} \dot{P} &= AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP, \\ K &= PC^T(NN^T)^{-1} \end{aligned}$$

Still, how do we choose  $M, N$  ??

## Back to the example: IEKF structure

$$\frac{d}{dt} \hat{q} = \hat{q} * (\omega_m - \hat{\omega}_b) + \hat{q} * (K_q E)$$

$$\frac{d}{dt} \hat{V} = A + \frac{1}{\hat{a}_s} \hat{q} * a_m * \hat{q}^{-1} + \hat{q} * (K_V E) * \hat{q}^{-1}$$

$$\frac{d}{dt} \hat{\omega}_b = K_\omega E$$

$$\frac{d}{dt} \hat{a}_s = \hat{a}_s K_a E,$$

$$E = \rho_{\hat{x}^{-1}} \begin{pmatrix} \hat{y}_V \\ \hat{y}_B \end{pmatrix} - \rho_{\hat{x}^{-1}} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} \hat{q}^{-1} * (\hat{V} - y_V) * \hat{q} \\ \hat{q}^{-1} * B * \hat{q} - y_B \end{pmatrix}.$$

The invariant state error  $g^{-1} \diamond \hat{g}$  reads

$$\begin{pmatrix} \mu \\ \nu \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} q^{-1} * \hat{q} \\ q^{-1} * (\hat{V} - V) * q \\ \hat{\omega}_b - \omega_b \\ \frac{\hat{a}_s}{a_s} \end{pmatrix},$$

hence we recover the quaternion error used in the MEKE. 

## Back to the example: Features of the IEKF

**Symmetry-preserving structure** rotations, translations and scaling in the appropriated frames leave the error system unchanged, which is meaningful from an engineering point of view.

**Sound geometric structure for the quaternion estimation equation:** it preserves the unit norm of the estimated quaternion.

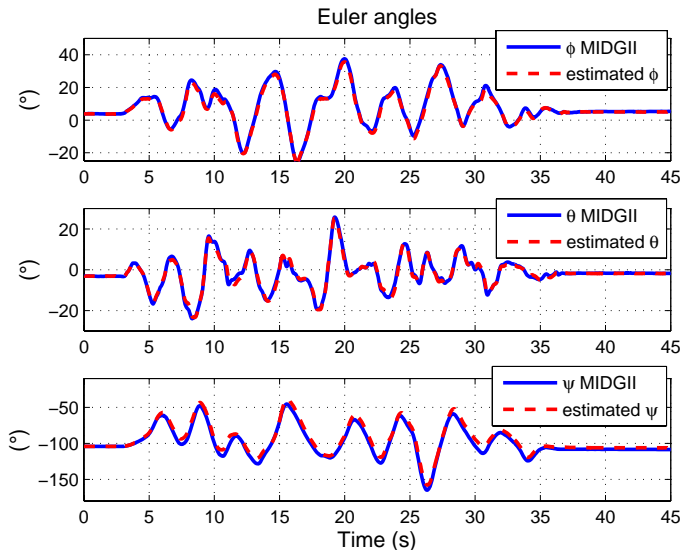
**Larger expected domain of convergence**

**Proposition:** the matrices  $A$  and  $C$  used for computing the gain matrix  $K$  are **constant** not only in level flight but also on a large set of trajectories (uniform acceleration, rotation with constant angular velocity...).

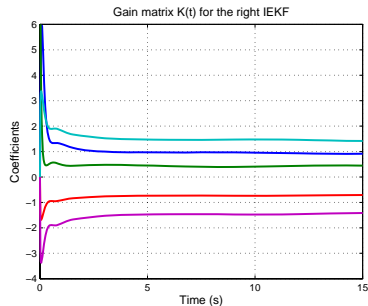
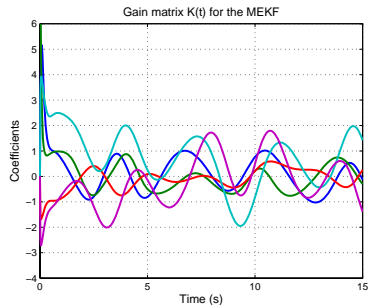
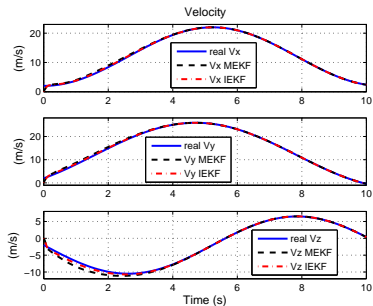
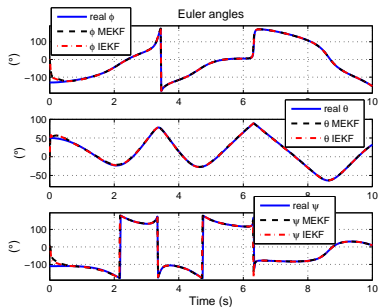
$$\text{Recall } \frac{d}{dt} \delta \eta = (A - KC) \delta \eta$$

# Back to the example: Numerical results

Experiment: estimated Euler angles. Comparison with a commercial INS-GPS device MIDG2.



# Simulation results: comparison of MEKF and IEKF



## IEKF: what about the noise matrices $M, N$ ?

The system with noise turned off

$$\begin{aligned}\frac{d}{dt}g &= f(g, u) + M(g)w \\ y &= h(g, u) + N(y)v\end{aligned}$$

is **invariant to the transformation**

$$g_1 = g_0 \diamond g, \quad u_1 = \psi_{g_0}(u), \quad y_1 = \rho_{g_0}(y)$$

It seems logical that the system with noise be **invariant** as well  
i.e.

$$\begin{aligned}\frac{d}{dt}g &= f(g_1, u_1) + M(g_1)w \\ y &= h(g_1, u_1) + N(y_1)v\end{aligned}$$

In particular we take  $\frac{d}{dt}\hat{g} = f(\hat{g}, u) + DL_{\hat{g}}Mw$ . On the example it yields the same driving noise as for the MEKF.

## IEKF: what about the noise matrices $M, N$ ?

With this definition of the noise matrices, the linearized error equation is a stochastic **multiplicative** linear differential equation

$$\frac{d}{dt}\delta\eta = (A - KC)\delta\eta + Q_1(\delta\eta, M(e)w) + Q_2(\delta\eta, KN(e)v),$$

It is NOT the linear model for which the KF is built

$$\frac{d}{dt}\delta\eta = (A - KC)\delta\eta - M(e)w + KN(e)v.$$

**Proposition:** For both equations, the mean and covariance of the  $\delta\eta$  are the same up to second order terms in the noise amplitude

Thus  $M, N$  can be chosen on the non-linear system, then build a Kalman filter with on the linearized system etc.

# Conclusion

- ▶ When a system possesses symmetries, compelling the EKF to preserve them is a way to provide it with the **rich geometric structure of the physical system**.
- ▶ On a group or a manifold more logical than the usual EKF based on the **linear** error  $\hat{x} - x$
- ▶ Corresponds to a time-invariant linear Kalman filter around a whole set of trajectories
- ▶ Particularly suited to aerospace (UAVs) applications, (Symmetries = Galilean invariances).