Left-invariant extended Kalman filter and attitude estimation

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***Abstract*— We consider a left-invariant dynamics on a Lie group. One way to define driving and observation noises is to make them preserve the symmetries. We propose a left-invariant (i.e, intrinsic and thus symmetry-preserving) extended Kalman filter such that the left-invariant estimation error obeys a stochastic differential equation independent of the system trajectory. The theory is illustrated by an attitude estimation example.**

I. Introduction

Consider the problem of estimating a process $x(t)$ from observation of a related process $y(t)$. This can be formulated in stochastic terms as a nonlinear filtering problem described by the stochastic differential equations

$$\dot{x} = f(x,t) + M(x)w \tag{1}$$

$$y = h(x) + N(x)v \tag{2}$$

$$x(t_0) = x^0 \tag{3}$$

The state process and the output process evolve on n and p dimensional manifolds. x^0 is gaussian, w and v are independent standard white gaussian driving and observation noises.

The nonlinear filtering problem is to compute in real time the conditional mean of the current state $x(t)$ given the past observations $\{y(s),\ t_0 < s < t\}$. The extended Kalman filter (EKF) yields an approximation of the conditional mean described by a finite number of statistics. They evolve according to ordinary differential equations (instead of stochastic partial differential equations) and thus can be computed in real time. Indeed the Extended Kalman Filter is an observer, and it is the most widely used non-linear filter. If we let $A(t) = \frac{\partial f}{\partial x}(\hat{x}(t))$ and $C(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$ (and MM^T, NN^T represent the covariances matrices of the driving and observation noise) the EKF equations write

$$\frac{d}{dt}\hat{x} = f(\hat{x},t) + K(t)(y - h(\hat{x})) \tag{4}$$

$$K(t) = P(t)C^T(t)(NN^T)^{-1} \tag{5}$$

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) + MM^T - P(t)C^T(t)(NN^T)^{-1}C(t)P(t) \tag{6}$$

The principle is to build a Kalman filter for the linear approximation of (1)-(2)-(3) and implement it on the nonlinear model. The EKF system does not take into account the specific geometry of the dynamics model (1). When the model admits symmetries, the EKF equations do not respect these symmetries. Usually symmetries are derived from physical considerations (independence of physical units, invariance by a change of coordinates...) and we would like the observer to respect these symmetries. In this paper, for the sake of simplicity, we will consider the (simple) case where the dynamics f admits n symmetries. This means a n-dimensional lie group acts on a n-dimensional state space. Under some assumptions on the regularity of the group action it is equivalent to consider x evolves on a Lie group and $\dot{x} = f(x,t)$ is a left-invariant dynamics. In this case we will see how to define the noises $M(x)dw$ and $N(x)dv$ so that (i) they preserve the symmetries (ii) the EKF takes into account the specific geometry of the system and preserves the symmetries. Our observer is given by (11)-(12)-(13)-(14). As far as we know such left-invariant (and thus intrinsic) formulation of an extended Kalman filter is new.

The idea of the intrinsic EKF is the following. The noises depend on the state variables in such a manner that the two following methods are equivalent:

1) Take the non-linear observer (11)-(12)-(13)-(14); consider the well-posed error (i.e, formulated intrinsically) between the estimated system and the true system with the noise turned off; compute its time derivative; linearize it; add additive noises to the linearized system, make a Kalman Filter for this system (first-order error equation); derive the gain K of the observer
2) take the dynamics (7)-(8)-(9) where the noises are left invariant; compute the well-posed error equation; linearize it; make a Kalman filter for the linearized error and derive the gain K.

There have been several attempts to introduce geometry in the problem of nonlinear filtering ([5], [10], [7]). Building observers which respect the geometry of the system have been done in [4], [2], [1], [9], [8], [6], [3]. The contribution of this paper is to make an intrinsic extended Kalman filter for a nonlinear left-invariant dynamics on a Lie group. The paper does not exhibit any convergence result. We only deal with the structure of the observer. Results about convergence of the EKF can be found, e.g., in [11]. Results about the convergence of left-invariant observers on a Lie group can be found in [4]. In section II we show how to define the noises which respect the geometry of the system. In section III we exhibit a left-invariant extended Kalman filter on a Lie group. In section IV we present a a simplified example of inertial navigation (only attitude estimation) also considered in [8]. Simulations are made in section V.

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II. Symmetries and noises

From now on we will consider the dynamics

$$\dot{x} = f(x,t) + M(x)w \tag{7}$$

$$y = h(x) + N(h(x))v \tag{8}$$

$$x(t_0) = x^0 \tag{9}$$

where x lies in a n-dimensional real Lie group G, $y \in \mathbb{R}^p$, x^0 is gaussian, w and v are independent standard white gaussian driving and observation noises, $M(x)$ is a morphism which depends on the parameter x and maps the tangent space at the identity element e of the group to the tangent space at x. $N(h(x))$ is an operator of $\mathbb{R}^p$. Moreover we will assume the model is such that f is a left invariant vector field:

$$\text{for all } g, x \ \in G \quad f(L_g x, t) = DL_g f(x,t) \tag{10}$$

where $L_g(x) = gx$ denotes the left multiplication on G and (as in the sequel) D stands for differentiation. The property also reads: for any $g \in G$ the system is left unchanged by the transformation $Z = L_g(z)$. Indeed (10) implies $\dot{Z} = f(Z,t)$. A good choice of the noise matrices when one deals with applications can be difficult to find. A logical way to define the noises in our case is to use geometrical considerations.

A. Invariant driving noise

Consider the model dynamics with noise turned off $\dot{z} = f(z,t)$. We will look at w as an aleatory element of the Lie algebra $\mathcal{G}$ of G, i.e, the tangent space to the group identity element e. To define a noise which preserves the symmetry we will assume $M(x)$ is defined by

$$M(x) = DL_x \ M(e)$$

where $M(e)$ is any endomorphism of $\mathcal{G}$. It implies for any $g \in G$, $DL_g M(x) = M(L_g(x))$ since $DL_g DL_x = DL_{gx}$. So the dynamics (7) affected by white noise is also invariant. Indeed let $X = L_g(x)$ we have

$$\begin{aligned}\dot{X} = DL_g \dot{x} &= DL_g(f(x,t) + M(x)w) \\ &= f(L_g(x),t) + DL_g M(x)w \\ &= f(X,t) + M(X)w\end{aligned}$$

which proves invariance of the dynamics (7). $M(e)$ represents the noise magnitude when x is close to e.

B. Invariant observation noise

We suppose that the output map $h : G \mapsto \mathbb{R}^p$ preserves the symmetries in the sense that it is G-equivariant (as in [1], [3], [4]). This means there exists a set of diffeomorphisms ρ_g parameterized by the group elements g such that for any $g_1, g_2 \in G$ $\rho_{g_1}\rho_{g_2} = \rho_{g_1 g_2}$ and for all $g, x \ \in G$ $h(L_g x) = \rho_g h(x)$. As in usual nonlinear filtering (equation (2)), we consider the output y to be the measurement of $h(x)$ affected by an additive white gaussian noise (due for example to the imperfections of the sensors). If the dynamics is invariant under a certain transformation (change of coordinates etc.) we want the observation noise to preserve this invariance in the following sense: $N(h(x))$ is a matrix depending on $h(x)$ which verifies for all $x, g \in G$

$$N(\rho_g(h(x))) = D\rho_g \ N(h(x))$$

where $D\rho_g$ denotes the tangent map induced by ρ_g. Thus the output Y at $X = L_g(x)$ is a function of g, $h(x)$ and the noise $N(h(x))v$

$$Y = h(X) + N(h(X))v = \rho_g(h(x)) + D\rho_g \ N(h(x))v$$

Moreover, we assume that the diffeomorphisms ρ_g are linear, as in the example of section IV, as well as in the examples treated in [4], [2], [1], [8], [6]. Thus $\rho_g(h(x)+N(h(x))v) = \rho_g(h(x)) + D\rho_g \ N(h(x))v$ and

$$Y = \rho_g(y)$$

where we recall Y is the output at $X = L_g(x)$. It means observation noise is such that the output does not destroy the symmetries in the sense it is G-equivariant (in other words compatible with the group action via ρ_g).

III. Intrinsic Extended Kalman Filter

Consider the dynamics (7)-(8)-(9). Let

$$A(t) : \mathcal{G} \ni \xi \mapsto [\xi, f(e,t)] \in \mathcal{G} \qquad C = Dh(e)$$
$$N = N(h(e)) \qquad M = M(e)$$

where e denotes the group identity element and [,] the Lie bracket associated to the Lie algebra $\mathcal{G}$. Consider the observer

$$\dot{\hat{x}} = f(\hat{x},t) + DL_{\hat{x}} K(t) \big(\rho_{\hat{x}^{-1}}(y) - \rho_{\hat{x}^{-1}}(h(\hat{x}))\big) \tag{11}$$

$$K(t) = P(t)C^T(t)(NN^T)^{-1} \tag{12}$$

$$\begin{aligned}\dot{P}(t) = {} & A(t)P(t) + P(t)A^T(t) + MM^T \\ & - P(t)C^T(t)(NN^T)^{-1}C(t)P(t)\end{aligned} \tag{13}$$

$$P(0) = P^0 \tag{14}$$

This observer is a sort of extended Kalman filter for (7)-(8)-(9) when driving and observation noises are chosen as explained in section II. Moreover it possesses the same symmetries as the system (7)-(8)-(9), i.e, it is left-invariant. Contrarily to the usual EKF the gain $K(t)$ depends on t but does not depend on the estimation $\hat{x}(t)$! It reminds the linear nonstationary case. It can be computed off-line. The form of the observer is inspired from [4], [3].

A. The observer is symmetry-preserving

An observer $\dot{\hat{x}} = F(\hat{x}, y)$ for (7)-(8)-(9) is said to be invariant (i.e, symmetry-preserving) (see [4], [3]) if it respects the geometry of the system in the following sense: for any $g \in G$ if we let $Y = \rho_g(y)$ and $\hat{X} = L_g(\hat{x})$ we have $\frac{d}{dt}\hat{X} = F(\hat{X}, Y)$. We are going to prove that the observer we propose is invariant by left multiplication. Indeed consider now (11). For any $g \in G$ if we let $L_g(x) = gx = X$ and $Y = \rho_g(y)$ we have

$$\begin{aligned}\frac{d}{dt}\hat{X} &= DL_g f(\hat{x},t) + DL_g DL_{\hat{x}} K(t) \big(\rho_{\hat{x}^{-1}}(y) - \rho_{\hat{x}^{-1}}(h(\hat{x}))\big) \\ &= f(g\hat{x},t) + DL_{g\hat{x}} K(t) \big(\rho_{(g\hat{x})^{-1}}(\rho_g(y)) - \rho_{(g\hat{x})^{-1}}(h(g\hat{x}))\big) \\ &= f(\hat{X},t) + DL_{\hat{X}} K(t) \big(\rho_{\hat{X}^{-1}}(Y) - \rho_{\hat{X}^{-1}}(h(\hat{X}))\big)\end{aligned}$$

since $DL_{g_1} f(g_2, t) = f(g_1 g_2, t)$, $\rho_{(g\hat{x})^{-1}} = \rho_{\hat{x}^{-1}}\rho_{g^{-1}}$, $h(g\hat{x}) = \rho_g h(\hat{x})$.

B. The observer is an intrinsic Extended Kalman Filter

The idea of the EKF is to linearize equations (7)-(8) around the trajectory $\hat{x}(t)$, then build a Kalman filter for the linear model and implement it on the non linear model. Then one can tell P is an approximation of $\mathbb{E}((\hat{x}-x)(\hat{x}-x)^T)$.

The idea of the intrinsic EKF for a left-invariant dynamics is to compute the time derivative of an intrinsic error, then linearize it considering the error is small, and build a Kalman filter for the linear error equation and implement it on the non linear model.

1) Intrinsic error: The usual state error $\Delta x = \hat{x} - x$ does not have any geometrical sense for $x, \hat{x} \in G$. One should rather use the left invariant equivalent state error $\eta = x^{-1}\hat{x}$ so that $\hat{x} = x\eta$ and not $\hat{x} = x + \Delta x$. This error is intrinsic in the sense that it is invariant under a left multiplication. Indeed for $g \in G$ we have $(gx)^{-1}(g\hat{x}) = x^{-1}\hat{x}$. When the observer is defined by (11)-(12)-(13)-(14), inspiring from [4] we have the following error equation

$$\begin{aligned} \frac{d}{dt}\eta &= DL_\eta f(e,t) - DR_\eta(f(e,t) + M(e)w) \\ &\quad + DL_\eta K\big(h(\eta^{-1}) + N(h(\eta^{-1}))v - h(e)\big) \end{aligned} \tag{15}$$

where R_η denotes the right multiplication by η on G. The computations are analogous to those detailed in section IV. A remarkable feature is that the non linear error equation only depends on the noises w and v, the time t, and η itself. It is thus independent of the trajectory and only depends on the relative positions of $\hat{x}$ and x, as in the linear case.

2) Linearized error equation: We want to make a Kalman filter for the linear model for the error. For a small error, i.e, η close to e, one can set $\eta = \exp(\xi)$ where ξ is a small element of the Lie algebra $\mathcal{G}$. Up to order 2 terms in $\|\xi\|$, we have the following linearized invariant state error equation on $\mathcal{G}$

$$\begin{aligned} \frac{d}{dt}\xi &= [\xi, f(e,t)] - (Id + r(\xi))M(e)w \\ &\quad + (Id + l(\xi))K\big((Id - d(\xi))N(h(e))v - Dh(e)\xi\big) \end{aligned} \tag{16}$$

where Id is the identity operator, l, r and d are the derivative at the identity of the group of DL, DR and $D\rho$: $l(\xi) = \frac{d}{ds}DL_{exp(s\xi)}$, $r(\xi) = \frac{d}{ds}DR_{exp(s\xi)}$ and $d(\xi) = \frac{d}{ds}D\rho_{exp(s\xi)}$. Such a stochastic differential equation is called a multiplicative equation. A treatment of multiplicative equations can be found in [12]: Let α be a parameter measuring the magnitude of the noises. Consider the solution of the following equation (17). Its mean and its covariance matrix are solutions to order α^2 of the differential equations verified by the mean and the covariance of the solution ξ of (16). Thus: if the observer made for (17) is robust, the expectation of ξ (verifying (16)) will also tend to 0 as $t \to \infty$; and the covariance matrix of the solution of (17) will be an approximation of the covariance of the solution of (16) to second order in α. Thus the solution of (17) yields an approximation of (16). The linearized error equation can be approximated now by the error equation:

$$\frac{d}{dt}\xi = [\xi, f(e,t)] - M(e)w + K\left(N(h(e))v - Dh(e)\xi\right) \tag{17}$$

when A, C, M and N are defined as above, this equation can be re-written

$$\frac{d}{dt}\xi = (A - KC)\xi - Mw + KNv \tag{18}$$

which is the usual linear equation for the error when one makes a linear Kalman filter. A Kalman filter for the linear model is given by equations (18) and (12)-(13)-(14) where P denotes the (intrinsic) covariance matrix $P(t) = \mathbb{E}(\xi\xi^T)$ of the linearized error. Thus (11)-(12)-(13)-(14) can be looked at as a left-invariant intrinsic EKF.

IV. Example

A. Quaternions

A quaternion $p \in \mathbb{H}$ can be viewed at a set of a scalar $p_0 \in \mathbb{R}$ and a vector $\vec{p} \in \mathbb{R}^3$,

$$p = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix}.$$

The quaternion multiplication $*$ writes

$$p * q := \begin{pmatrix} p_0 q_0 - \vec{p}\cdot\vec{q} \\ p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q} \end{pmatrix}.$$

The identity element is $e := \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$ and $(p*q)^{-1} = q^{-1} * p^{-1}$. Any vector $\vec{p} \in \mathbb{R}^3$ can be looked at as a quaternion $p := \begin{pmatrix} 0 \\ \vec{p} \end{pmatrix}$ and we will systematically identify vectors $\vec{p} \in \mathbb{R}^3$ with the corresponding quaternion $p \in \mathbb{H}$. To any quaternion q whose norm is 1 one can associate a rotation matrix $R_q \in SO(3)$ the following way: $q^{-1} * \vec{p} * q = R_q(\vec{p})$ for all $\vec{p}$. A motion on the group of rotations of three-dimensional euclidian space can be written thanks to quaternions or rotation matrices. The use of quaternion is standard in inertial navigation problems. Quaternions with norm 1 are suited to computer simulations because it is easier to maintain numerically the norm of q equal to 1 rather than a rotation matrix in $SO(3)$.

B. An attitude filtering problem

The motion equations of kinematics for a flying rigid body write (using quaternions $\mathbb{H}$)

$$\dot{q} = \frac{1}{2} q * \omega + q * Mw \tag{19}$$

where $q \in \mathbb{H}$ is the quaternion of norm one which represents the rotation which maps the earth frame to the body frame, ω(t) is the instantaneous angular velocity vector measured by gyroscopes, M is a constant 3×3 matrix and w is a standard gaussian white noise of $\mathbb{R}^3$, $*$ is the non commutative quaternion multiplication.

The measurements are the instantaneous rotation vector $\omega(t)$, as well as the magnetic field and the specific acceleration. The two last measurements (made by magnetometers and accelerometers) are combined together to provide an algebraic estimation q_y of the attitude q. This approximation is valid when the acceleration of the body is small compared to the magnitude of the earth magnetic field (10 $m.s^{-2}$). The method to get q_y is explained in [8] which considers the same dynamics (19) with the same output q_y but without any noise ($M = N = 0$). We consider that the driving noise $q * Mw$ represents the error made making this approximation (more practically it means that we suppose error sources arising from gyro can be modelled white noise). The output of the system is

$$q_y = q + q * Nv$$

where v is a standard gaussian noise of $\mathbb{R}^3$ which corresponds to the imperfections of the sensors, and N a real 3×3 matrix.

C. Symmetries of the system

The kinematics equations (19) are independent of the choice of the earth frame. Notice there is no reason why the driving and observation noise should depend on the coordinates either. They were chosen so that they respect this invariance under a change of coordinates. We want to make an intrinsic extended Kalman filter for this problem, i,e, which does not depend on the choice of the earth frame and hence preserves the symmetries which is an intrinsic property .

Let us verify the dynamics is a left-invariant dynamics on a Lie group. The subgroup of quaternions with norm 1 is a Lie group. Let r be any quaternion with norm 1. Make the left multiplication $Q = r * q$. The equations for Q are similarly written

$$\begin{aligned} \dot{Q} &= \frac{1}{2} r * q * \omega + r * q * Mw \\ &= \frac{1}{2} Q * \omega + Q * Mw \end{aligned}$$

Moreover the output is compatible since

$$Q_y = r * q_y = r * (q + q * Nv) = Q + Q * Nv$$

The physical sense is the following : r is a quaternion whose norm is 1 so it represents a rotation and $Q = r * q$ represents the attitude quaternion after having made the earth frame rotate. The dynamics and output equation correspond to (7)-(8) when noises are chosen as in section II.

D. An intrinsic extended Kalman filter

Instead of considering the usual linear error $\hat{q} - q$ we consider the equivalent intrinsic error $\eta = q^{-1}\hat{q}$ (see section III-B.1). η is a quaternion which corresponds to the rotation which maps the true frame to the estimated frame and hence does not depend on the choice of the earth frame. P(0) is a definite positive matrix representing the initial covariance of the error $\mathbb{E}(\eta\eta^T)$. The commutator $p \mapsto \frac{1}{2}(p * \omega(t) - \omega(t) * p)$ is a linear function of the quaternion p we let $A(t)$ denote. Consider the following observer (corresponding to (11) -(12)-(13)):

$$\begin{aligned} \frac{d}{dt}\hat{q} &= \frac{1}{2}\hat{q} * \omega + \hat{q} * K(t)(\hat{q}^{-1} * q_y - 1) \\ K(t) &= P(t)(NN^T)^{-1} \\ \dot{P}(t) &= A(t)P(t) + P(t)A^T(t) + MM^T \\ &\quad - P(t)(NN^T)^{-1}P(t) \end{aligned} \tag{20}$$

The gain P(t) is calculated so that the tangent approximation of the error equation verifies the linear Kalman filter equations. Indeed the estimation error equation has the following dynamics (corresponding to (15)):

$$\begin{aligned} \dot{\eta} = &\frac{1}{2}(\eta * \omega - \omega * \eta) - Mw * \eta \\ &+ \eta * K(\eta^{-1} - 1 + \eta^{-1} * Nv) \end{aligned}$$

since $\frac{d}{dt}q^{-1} = -q^{-1} * \dot{q} * q^{-1} = -\omega * q^{-1}$ and $\hat{q}^{-1} * q = \eta^{-1}$. Let us assume the $\hat{q}$ is close to q and set $\eta = 1 + \delta\eta$ with $\delta\eta$ a small quaternion whose first coordinate is equal to 0 ,i.e $\delta\eta$ belongs to the tangent space at the identity of the subgroup of quaternions whose norm is 1. We have up to second order terms in $\delta\eta$

$$\begin{aligned} \dot{\delta\eta} = &A(t)\delta\eta - Mw - Mw * \delta\eta \\ &- \delta\eta * KNv + KNv - K\delta\eta \end{aligned}$$

Let us suppose the noise Mw and Nv are small enough so that $Mw * \delta\eta$ and $\delta\eta * KNv$ are "second order terms". If the noises are too big, there should be no reason why η can remain small. The linear approximation of the error equation becomes (corresponding to (17)) :

$$\dot{\delta\eta} = A(t)\delta\eta - K\delta\eta - Mw + KNv$$

Thus filter (20) is an intrinsic observer such that the gains correspond to a Kalman filter built for linear approximation of the error. This is the same principle as the usual EKF but here the error is expressed in an intrinsic way. As an interesting by-product, one can notice the error equation does not depend on $\hat{q}(t)$ ([4], [3]) and thus the convergence (and the tuning of the gains i,e, the choice of M and N) does not depend on the trajectory although the dynamics is nonlinear.

V. Simulations

To get realistic values of ω, and q_y we generated a trajectory of a VTOL-type drone which the same as the one in [4]. The center of mass of the flying body follows a circle whose radius is 5 meters and stops. The following high frequencies are added to signals corresponding to the smooth trajectory of the VTOL-type drone $\omega(t)$ and $q(t)$ in order to represent the imperfections of the sensors : $q_y(t) = q(t) * (1 + .2\ \sigma_1)$ and for the measured $\omega(t)$ we take $\omega(t) + .5\ \sigma_2$, where the σ_i are independent normally distributed random 3-dimensional vectors with mean 0 and variance 1. The amplitude of the noise is 20% of the maximal value of the variables.

The initial conditions are such that the initial rotation differs from the true one up to a $2\pi/3$ angle. It writes for the corresponding quaternions:

$$\begin{aligned} q(0) &= [1, 0, 0, 0] \\ \hat{q}(0) &= [\cos(\pi/3), \sin(\pi/3)/\sqrt{(}3), \\ &\quad - \sin(\pi/3)/\sqrt{(}3), \sin(\pi/3)/\sqrt{(}3)] \end{aligned}$$

We take for the noises matrices : $M = 0.5 * I_3$ and $N = 0.2 * I_3$ and $P(0) = 0.1 * I_3$ (for instance) where I_3 is the 3×3 identity matrix. The measurements and the behavior of $\hat{q}$ and q (as well as the estimation error) are represented by figures 1 and 2. Figure 3 shows the error convergence is the same when we take the same $\omega(t)$ and the same initial error $\eta(0)$ but a different initial $q(0)$.

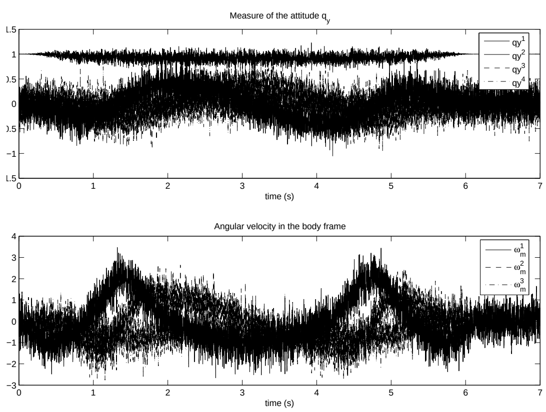

Fig. 1. Measured noisy signals : attitude q_y and angular velocity ω (rad/s) in the body frame.

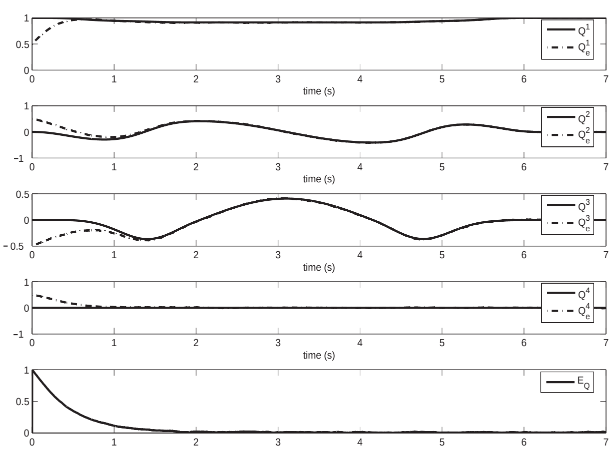

Fig. 2. Four coordinates of attitude q (solid line) and estimated attitude $\hat{q}$ (dashed-line); estimation error $E_Q = \|q^{-1}\hat{q} - 1\|$ using left-invariant EKF (20).

VI. Conclusion

The intrinsic EKF has two practical advantages : a systematic way of choosing noises hence of tuning the gains, and the error equation (hence the assignment of the eigenvalues) does not depend on the trajectory $x(t)$ but only on the relative positions of x and $\hat{x}$, although the dynamics is not linear. One can also consider the general case of a dynamics on a n-dimensional manifold invariant under the action of a r-dimensional Lie group with $r < n$, i.e, there are only r symmetries generated by local group actions. In this case it is also possible to define an invariant state-error [4] and to make an symmetry-preserving extended Kalman filter in a similar way to the one developed in this paper.

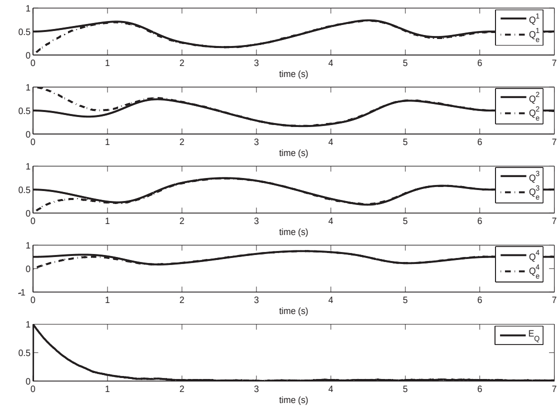

Fig. 3. Idem: four coordinates of attitude q (solid line) and estimated attitude $\hat{q}$ (dashed-line); estimation error $E_Q = \|q^{-1}\hat{q} - 1\|$ using left-invariant EKF (20). The only difference with Fig 2 is that the initial condition $q(0) = [0.5, 0.5, 0.5, 0.5]$ is different, but $\eta(0)$ is the same. Although the trajectory $q(t)$ is different, the error E_Q has the same behavior.

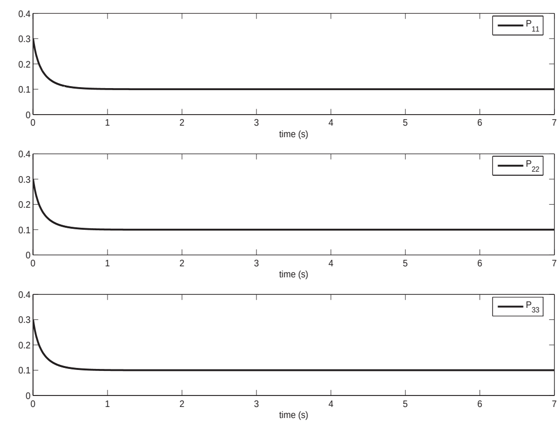

Fig. 4. Diagonal coefficients of the matrix P. Every parameter is the same as in section V except that here we took $P(0) = 0.3 * I_3$ to illustrate the convergence of the diagonal coefficients.

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