Symmetry preserving observers

The observation problem:

Consider a nonlinear system with dynamics described by

$$\frac{d}{dt}x(t) = f(x(t), u(t))$$

equipped with sensors yielding measurements

$$y(t) = h(x(t), u(t))$$

with $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathcal{Y} \subset \mathbb{R}^p$, $p \leq n$, and $u(t)$ is a known input (control, constant parameter, t ..)

Ideal observation problem:

- Observer state z in the same space as system state x
- The evolution of $\hat{x}(t)$ is given by :

$$\frac{d}{dt}\hat{x}(t) = F(\hat{x}(t), y(t), u(t))$$

Symmetries in physical models (invariances)

- A thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation
- How can a model be "symmetrical" ?
- What operation can we do to a an experiment, and leave the result the same ?

Symmetries in physical models (invariances)

- A thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation
- How can a model be "symmetrical" ?
- What operation can we do to a an experiment, and leave the result the same ?

Exothermic chemical reactor:

$$\begin{aligned}
\frac{d}{dt}x_1 &= D(x_1^{in} - x_1) - k_0 \exp(-E/RT)x_1 \\
\frac{d}{dt}T &= D(T^{in} - T) + \alpha \Delta H \exp(-E/RT)x_1 + v \\
y(t) &= T.
\end{aligned}$$

Change of units (mol→ kg): $X_1^{in} = Mx_1^{in},\ X_1 = Mx_1$, $A = \alpha/M$?

Symmetries in physical models (invariances)

- A thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation
- How can a model be "symmetrical" ?
- What operation can we do to a an experiment, and leave the result the same ?

Exothermic chemical reactor:

$$\begin{aligned}
\frac{d}{dt}x_1 &= D(x_1^{in} - x_1) - k_0 \exp(-E/RT)x_1 \\
\frac{d}{dt}T &= D(T^{in} - T) + \alpha \Delta H \exp(-E/RT)x_1 + v \\
y(t) &= T.
\end{aligned}$$

Change of units (mol→ kg): $X_1^{in} = Mx_1^{in},\ X_1 = Mx_1,\ A = \alpha/M$?

$$\begin{aligned}
\frac{d}{dt}X_1 &= D(X_1^{in} - X_1) - k_0 \exp(-E/RT)X_1 \\
\frac{d}{dt}T &= D(T^{in} - T) + A \Delta H \exp(-E/RT)X_1 + v \\
y(t) &= T.
\end{aligned}$$

Symmetries: Transformation groups

Let G be a Lie group with dim $G \leq$ dim $\mathcal{X}$. We consider the following *transformation group* acting on $\mathcal{X} \times \mathcal{U} \times \mathcal{Y}$: each $g \in G$ induces

- $\varphi_g, \psi_g, \rho_g$ diffeomorphisms of (resp.) $\mathcal{X}, \mathcal{U}, \mathcal{Y}$
- $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 \cdot g_2}\ , \quad \psi_{g_1} \circ \psi_{g_2} = \psi_{g_1 \cdot g_2}\ , \quad \rho_{g_1} \circ \rho_{g_2} = \rho_{g_1 \cdot g_2}$

For instance S^1 acting on the plane: $R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1 + \theta_2}$.

For any $g \in G$ consider the transformation

$$(X, U, Y) = \big(\varphi_g(x), \psi_g(u), \varrho_g(y)\big)$$

Definition The system $\frac{d}{dt}x = f(x, u)$ is *S-invariant* if for all g, x, u

$$\frac{d}{dt}X = f(X, U)$$

Definition The output $y = h(x, u)$ is *equivariant* if for all g, x, u

$$Y = h(X, U)$$

Symmetries and observers

In the chemical reactor example, $x = (x_1, T)$, $u = (x^{in}, \alpha)$, $y = T$.

The transformation group is made of scalings:
$G = \mathbb{R}_+^*,\ \varphi_g(x1, T) = (gx_1, T),\ \psi_g(x^{in}, \alpha) = (gx^{in}, \alpha/g),\ \rho_g(T) = T$.
It leaves the system unchanged

$$\frac{d}{dt}X = f(X, U),\ \ Y = h(X, U)$$

“Popular" observers with correction term linear in the output error write

$$\begin{aligned}
\tfrac{d}{dt}\hat{x}_1 &= D(x_1^{in} - \hat{x}_1) - k_0 \exp(-E/R\hat{T})x_1 + L_1(t)(\hat{T} - T)\\
\tfrac{d}{dt}\hat{T} &= D(T^{in} - \hat{T}) + \alpha \Delta H \exp(-E/R\hat{T})\hat{x}_1 + v + L_2(t)(\hat{T} - T)
\end{aligned}$$

Symmetries and observers

In the chemical reactor example, $x = (x_1, T)$, $u = (x^{in}, \alpha)$, $y = T$.

The transformation group is made of scalings:
$G = \mathbb{R}_+^*,\ \varphi_g(x1, T) = (gx_1, T),\ \psi_g(x^{in}, \alpha) = (gx^{in}, \alpha/g),\ \rho_g(T) = T$.
It leaves the system unchanged

$$\frac{d}{dt}X = f(X, U),\ \ Y = h(X, U)$$

“Popular" observers with correction term linear in the output error write

$$\begin{aligned} \tfrac{d}{dt}\hat{x}_1 &= D(x_1^{in} - \hat{x}_1) - k_0 \exp(-E/R\hat{T})x_1 + L_1(t)(\hat{T} - T) \\ \tfrac{d}{dt}\hat{T} &= D(T^{in} - \hat{T}) + \alpha\Delta H \exp(-E/R\hat{T})\hat{x}_1 + v + L_2(t)(\hat{T} - T) \end{aligned}$$

WE HAVE NOT

$$\frac{d}{dt}\hat{X} = F(\hat{X}, U, Y)$$

where $\hat{X} = \varphi_g(x)$.

Symmetry-preserving observers for the class of S-invariant systems

Definition: The observer $\frac{d}{dt}\hat{x} = F(\hat{x}, u, y)$ is *S-invariant* if for all $g, \hat{x}, u, y$

$$\frac{d}{dt}\hat{X} = F(\hat{X}, U, Y)$$

-How can we build them ?

Definitions

- $I(x, u)$ is *an S-invariant* if $I(X, U) = I(x, u)$

- $E(\hat{x}, u, y)$ is an *S-invariant output error* if (analog of $\hat{y} - y$)
  - $y \mapsto E(\hat{x}, u, y)$ is invertible for all $\hat{x}, u$
  - $E\big(\hat{x}, u, h(\hat{x}, u)\big) = 0$ for all $\hat{x}, u$
  - $E\big(\hat{X}, U, Y\big) = E(\hat{x}, u, y)$ for all $\hat{x}, u, y$

-$W(\hat{x}) = \big(w_1(\hat{x}), .., w_n(\hat{x})\big)$ is an *S-invariant frame* if
  - It forms a basis of the tangent space at each x, and $\frac{d}{dt}x = w_i(x)$ is S-invariant for each i .

[Aghannan, Rouchon (CDC 2002)]

Structure of symmetry-preserving observers

Lemma: Every S-invariant candidate observer reads[1]

$$\frac{d}{dt}\hat{x} = f(\hat{x},u) + W(\hat{x})L\Big(I(\hat{x},u), E(\hat{x},u,y)\Big)E(\hat{x},u,y)$$

- $E(\hat{x},u,y)$ S-invariant output error
- $W(\hat{x}) = \big(w_1(\hat{x}),..,w_n(\hat{x})\big)$ S-invariant frame
- $I(\hat{x},u)$ S-invariant
- $L(I,E)$ freely chosen $n \times p$ gain matrix

---

[1] Bonnabel, Martin, Rouchon: Symmetry-preserving observers (IEEE-TAC, 2008).

Structure of symmetry-preserving observers

Lemma: Every S-invariant candidate observer reads[1]

$$\frac{d}{dt}\hat{x} = f(\hat{x},u) + W(\hat{x})L\Big(I(\hat{x},u), E(\hat{x},u,y)\Big)E(\hat{x},u,y)$$

- $E(\hat{x},u,y)$ S-invariant output error
- $W(\hat{x}) = \big(w_1(\hat{x}),..,w_n(\hat{x})\big)$ S-invariant frame
- $I(\hat{x},u)$ S-invariant
- $L(I,E)$ freely chosen $n \times p$ gain matrix

All symmetry-preserving observers write

$$\begin{aligned}
\tfrac{d}{dt}\hat{x}_1 &= D(x_1^{in}-\hat{x}_1) - k_0\exp(-E/R\hat{T})\hat{x}_1 + L_1\Big(I(\hat{x},u),\hat{T}-T\Big)(\hat{T}-T)\hat{x}_1 \\
\tfrac{d}{dt}\hat{T} &= D(T^{in}-\hat{T}) + \alpha\Delta H\exp(-E/R\hat{T})\hat{x}_1 + v + L_2\Big(I(\hat{x},u),\hat{T}-T\Big)(\hat{T}-T)
\end{aligned}$$

where $I(\hat{x},u) = (\hat{X}/X^{in}, \alpha\hat{x}, \hat{T}, T^{in})$.

---

[1] Bonnabel, Martin, Rouchon: Symmetry-preserving observers (IEEE-TAC, 2008).

Structure of symmetry-preserving observers

Lemma: Every S-invariant candidate observer reads[1]

$$\frac{d}{dt}\hat{x} = f(\hat{x},u) + W(\hat{x})L\Big(I(\hat{x},u), E(\hat{x},u,y)\Big)E(\hat{x},u,y)$$

- $E(\hat{x},u,y)$ S-invariant output error
- $W(\hat{x}) = \big(w_1(\hat{x}),..,w_n(\hat{x})\big)$ S-invariant frame
- $I(\hat{x},u)$ S-invariant
- $L(I,E)$ freely chosen $n \times p$ gain matrix

All symmetry-preserving observers write

$$\begin{aligned}
\tfrac{d}{dt}\hat{x}_1 &= D(x_1^{in} - \hat{x}_1) - k_0 \exp(-E/R\hat{T})\hat{x}_1 + L_1\Big(I(\hat{x},u), \hat{T}-T\Big)(\hat{T}-T)\hat{x}_1 \\
\tfrac{d}{dt}\hat{T} &= D(T^{in} - \hat{T}) + \alpha\Delta H \exp(-E/R\hat{T})\hat{x}_1 + v + L_2\Big(I(\hat{x},u), \hat{T}-T\Big)(\hat{T}-T)
\end{aligned}$$

where $I(\hat{x},u) = (\hat{X}/X^{in}, \alpha\hat{x}, \hat{T}, T^{in})$.

Remark: Note that $\hat{x}_1$ is AUTOMATICALLY positive !

---

[1] Bonnabel, Martin, Rouchon: Symmetry-preserving observers (IEEE-TAC, 2008).

So what?

Definition: $(\hat{x}, x) \mapsto \eta(\hat{x}, x) \in \mathcal{X}$ is an *S-invariant state error* if (analog of $\hat{x} - x$)

- $x \mapsto \eta(\hat{x}, x)$ is invertible for all $\hat{x}$ and vice versa
- $\eta(x, x) = 0$ for all x
- $\eta(\hat{X}, X) = \eta\big(\varphi_g(\hat{x}), \varphi_g(x)\big) = \eta(\hat{x}, x)$ for all $g, \hat{x}, x$

Lemma: The error system is “nearly" autonomous:

$$\tfrac{d}{dt}\eta = \Upsilon\big(\eta, I(\hat{x}, u)\big), \text{ where } \eta \text{ is an } S\text{-invariant state error}$$

and we have dim I =dim x+dim u- dim g. When dim $G =$ dim $\mathcal{X}$, autonomous error system but for the “free" known S-invariant I! Reminds the linear case $\frac{d}{dt}e = (A - LC)e$.

Definition: A *permanent trajectory* is defined by $I(x(t), u(t)) \equiv c$

i.e. the S-invariant I is constant over the time.

So what?

Around every permanent trajectory we have $\frac{d}{dt}\eta = \Upsilon(\eta, c)$.

Main benefit of the symmetry-preserving approach:

when $\dim G = \dim \mathcal{X}$, the observer is easily tuned for (at least) local convergence around every “permanent" trajectory, with a good local behavior (interesting practical property)

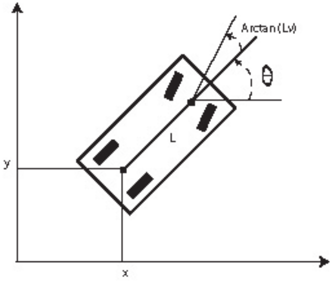
A tutorial exemple : non-holonomic car

Motion equations :

$$\frac{d}{dt}x = u\cos\theta$$
$$\frac{d}{dt}y = u\sin\theta$$
$$\frac{d}{dt}\theta = uv$$

Output :

$$h(x, y, \theta) = (x, y)$$



A tutorial exemple : non-holonomic car, $G = SE(2)$

Motion equations :

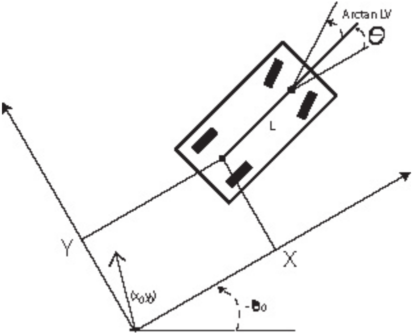
$$\frac{d}{dt}X = U\cos\Theta$$

$$\frac{d}{dt}Y = U\sin\Theta$$

$$\frac{d}{dt}\Theta = UV$$

Output :

$$h(X, Y, \Theta) = (X, Y)$$



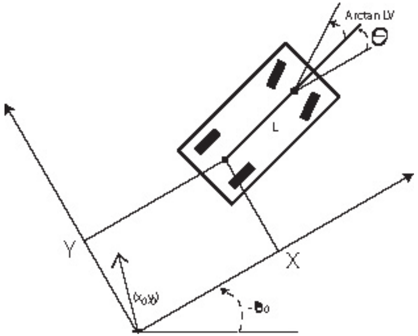
A tutorial exemple : non-holonomic car, $G = SE(2)$

Motion equations : **Output :**

$$\frac{d}{dt}X = U\cos\Theta \qquad\qquad h(X, Y, \Theta) = (X, Y)$$

$$\frac{d}{dt}Y = U\sin\Theta$$

$$\frac{d}{dt}\Theta = UV$$



Permanent trajectories: Lines and circles.

A tutorial exemple : non-holonomic car, $G = SE(2)$

Invariance by rotation and translation $g = (x_g, y_g, \theta_g) \in G$

$$\begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \\ \theta + \theta_g \end{pmatrix}$$

$$\varphi_{(x_g, y_g, \theta_g)}(x, y, \theta) = \begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}, \quad \psi_{(x_g, y_g, \theta_g)}(u, v) = \begin{pmatrix} u \\ v \end{pmatrix}$$

An S-invariant observer

$$\begin{pmatrix} \frac{d}{dt}\hat{x} \\ \frac{d}{dt}\hat{y} \\ \frac{d}{dt}\hat{\theta} \end{pmatrix} = \begin{pmatrix} u\cos\hat{\theta} \\ u\sin\hat{\theta} \\ uv \end{pmatrix} + \begin{pmatrix} \begin{array}{cc|c} \multicolumn{2}{c}{\multirow{2}{*}{$R_{-\hat{\theta}}$}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{array} \end{pmatrix} \cdot L \cdot R_{\hat{\theta}} \cdot \begin{pmatrix} \hat{x} - x \\ \hat{y} - y \end{pmatrix}$$

where L is a 3×2 gain matrix and $R_{\hat{\theta}} := \begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix}$

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u) + W(\hat{x})L\Big(I(\hat{x}, u), E(\hat{x}, u, y)\Big)E(\hat{x}, u, y)$$

A tutorial exemple : non-holonomic car, $G = SE(2)$

“Alternative" state error

$$\begin{pmatrix}\eta_x\\ \eta_y\end{pmatrix} = R_{\hat\theta}\cdot\begin{pmatrix}\hat x - x\\ \hat y - y\end{pmatrix} \qquad R_{\hat\theta} := \begin{pmatrix}\cos\hat\theta & \sin\hat\theta\\ -\sin\hat\theta & \cos\hat\theta\end{pmatrix}$$

$$\eta_\theta = \hat\theta - \theta \qquad E = \begin{pmatrix}\eta_x\\ \eta_y\end{pmatrix}$$

Autonomous error system but for the “free" known u, v

$$\frac{d}{dt}\begin{pmatrix}\eta_x\\ \eta_y\\ \eta_\theta\end{pmatrix} = \begin{pmatrix}u(1-\cos\eta_\theta) + (uv + L_{31}\eta_x + L_{32}\eta_y)\eta_y\\ u\sin\eta_\theta - (uv + L_{31}\eta_x + L_{32}\eta_y)\eta_x\\ 0\end{pmatrix} + L\cdot\begin{pmatrix}\eta_x\\ \eta_y\end{pmatrix}$$

Easy tuning on the linearized error system with $a, b, c > 0$.

$$\frac{d}{dt}\begin{pmatrix}\delta\eta_x\\ \delta\eta_y\\ \delta\eta_\theta\end{pmatrix} = \begin{pmatrix}uv\delta\eta_y\\ u\delta\eta_\theta - uv\delta\eta_x\\ 0\end{pmatrix} + \begin{pmatrix}-|u|a & -uv\\ uv & -|u|c\\ 0 & -ub\end{pmatrix}\begin{pmatrix}\delta\eta_x\\ \delta\eta_y\end{pmatrix}$$

Further properties: *S*-invariant dynamics and observers on a Lie group[2]

Let $L_{g_1}(g_2) = g_1 g_2$ denote the left multiplication on G.

In the car exemple $G = \mathcal{X}$ and we considered left-invariant dynamics on a Lie group G

$$\begin{aligned} \frac{d}{dt} g &= DL_g \omega_s(t) \quad (\text{Think about} \frac{d}{dt} R = R(\omega \wedge \cdot)) \\ y &= h(g) \end{aligned}$$

Now φ_g is replaced by L_g. If the output h is equivariant we have an explicit and intrinsic formula for *every* S-invariant observer

$$\frac{d}{dt}\hat{g} = DL_{\hat{g}}\omega_s(t) + DL_{\hat{g}} \left(\sum_{i=1}^{n} \mathcal{L}_i(\rho_{\hat{g}^{-1}}(y)) W_i \right)$$

with $(W_1, ..W_n)$ basis of the Lie algebra $\mathcal{L}_i(h(e))) = 0$

---

[2] Bonnabel, Martin, Rouchon : Non-linear Symmetry preserving observers on Lie groups. IEEE-TAC, June 2009.

The S-invariant state error is

$$\eta = g^{-1}\hat{g}$$

The error equation

$$\frac{d}{dt}\eta = DL_{\eta}\omega_s - DR_{\eta}\omega_s + DL_{\eta}\Big(\sum_{i=1}^{n} \mathcal{L}_i \circ h(\eta^{-1})\ W_i\Big)$$

reminds $\frac{d}{dt}e = (A(t) + LC)e$

What if the condition on the output is $h(g_1 g_2) = \rho_{g_2}(h(g_1))$?

- $\eta = \hat{g}g^{-1} \Rightarrow$ AUTONOMOUS error equation ![3]
- reminds $\frac{d}{dt}e = LCe$

Lemma: When, moreover, $\mathcal{Y}$ is a homogeneous space under the group action, one can design left-invariant observers such that $\hat{y}$ converges to y for almost all initial condition.[4]

---

[3] Bonnabel, Martin, Rouchon (CIFA 2006). Lageman, Trumpf, Mahony (MTNS 2008)

[4] Lageman, Trumpf, Mahony (ECC 2009)

A major application: multi-sensor fusion for small UAVs

For GPS/IMU data fusion for small UAV's, symmetry-preserving observers have many practical interests, making them a challenger to the usual popular observers (EKF...).

- There is an explicit method to design a candidate nonlinear observer.
- Many symmetries: Galilean invariances.
- Convergence around a whole set of trajectories: converges around *any* permanent trajectory.
- Easy to tune : Gain tuning on the linearized system around permanent trajectories.
- Low computational cost (implemented on a cheap (5$) 8-bit micro-controller)[5]

Some invariant observers used in data fusion for small UAVs. See, e.g.,

- Mahony, Hamel, Pflimlin (CDC 2005, IEEE-TAC 2008)
- Vasconcelos, Silvestre and Oliveira (CDC 2008)
- Martin, Salaun (CDC 2008)
- Bonnabel, Rouchon (Springer 2005)

---

[5] See PhD Thesis of Erwan Salaün where such kind of observers are extensively developed and tested experimentally on micro-controllers.

Conclusion

Invariance is just a way to "encode physics" in nonlinear estimation and filtering processes.

- Chemical reactors where the group is just associated to changes of units and invariance relies on the fact that material and energy balance equations do not depend on chosen units.
- Data fusion for small UAVs where the group is generaly associated with the Galilean invariances.
- Infinite dimensional systems described by PDE's such as the Saint-Venant equation[6].
- For data fusion between camera, acceleros, gyros and magnetos... Galilean invariance could certainly be exploited in a much more systematic way....

---

[6] D. Auroux and S. Bonnabel. Symmetry-preserving observers for some water-tank problems (Accepted in IEEE-TAC)