

# A simple feedback-loop for a frequency lock on a narrow atomic transition

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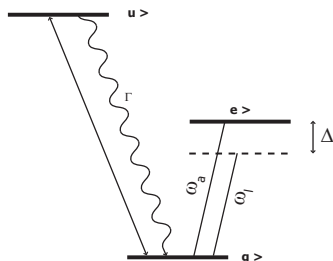
# The control problem

## Physical system:

Mono-atomic gaz.  $E_e - E_g = \hbar\omega_{atom}$ .

## Measurement:

- ▶ Photons emitted by the population of the ground state  $|g\rangle$ :  $|u\rangle$  decays quickly to  $|g\rangle$ , and population  $|u\rangle$  is negligible.  $\Delta$



## Model

$$i\frac{d}{dt}\Psi = \left( \frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x \right) \Psi, \quad \Psi = \begin{pmatrix} \Psi_g \\ \Psi_e \end{pmatrix} \in \mathbb{C}^2$$

- ▶  $\Psi$  is the wave function:  $|\Psi_g|^2 + |\Psi_e|^2 = 1$ .
- ▶  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  are the Pauli matrices
- ▶  $\sigma_x^2 = 1$ ;  $\sigma_x\sigma_y = i\sigma_z$ .

## Model

$$i \frac{d}{dt} \Psi = \left( \frac{\Delta}{2} \sigma_z + \frac{\Omega_R}{2} \sigma_x \right) \Psi, \quad \Psi = \begin{pmatrix} \Psi_g \\ \Psi_e \end{pmatrix} \in \mathbb{C}^2$$

## Parameters

- ▶ Laser detuning  $\Delta = \omega_{atom} - \omega_l$ .
- ▶ Laser amplitude  $\Omega_R / \mu$  with  $\mu > 0$ .
- ▶ We have  $1 \ll \Omega_R$  and  $\Delta \ll \Omega_R$ .

## Control problem

$$\begin{aligned} \frac{d}{dt} \Delta &= pu, \quad p > 0 \\ y &= 2|\Psi_g|^2 - 1 \end{aligned}$$

With control  $u$ ,  $\Omega_R$  find the atomic transition  $\omega_{atom}$ :

We want  $\Delta \rightarrow 0$ .

# Control law

$$i \frac{d}{dt} \Psi = \left( \frac{\Delta}{2} \sigma_z + \frac{\Omega_R}{2} \sigma_x \right) \Psi, \quad \Psi = \begin{pmatrix} \Psi_g \\ \Psi_e \end{pmatrix} \in \mathbb{C}^2$$

- ▶ set  $\Omega_R = \bar{\Omega}_R$  to a positive constant



$$u = Ky - \frac{K_2 K_f}{s + K_f} y$$

- ▶  $K, K_2, K_f > 0$  satisfy  $\Omega_R \gg K_f > \frac{pK}{\Omega_R}$  and  $\Omega_R^2 \gg pK, pK_2$

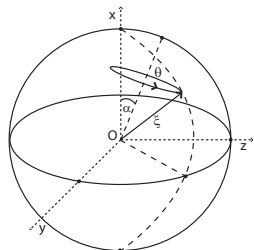
**Result** With such control we have locally

- ▶  $\Delta \rightarrow 0$ .

# Motivation: Atomic clocks

- ▶ Atoms in Dilute mono-atomic gases can be considered as perfect quantum systems with increasing energies  $E_i = \hbar\omega_i$  depending only of the atomic species considered.
- ▶ Atomic clocks take advantage of the universality of  $E_{eg} = \hbar\omega_{atom}$  to deliver a **stable**, periodic electromagnetic signal which frequency  $\omega_{atom}$  in the *GHz* range.
- ▶ The laser frequency lock  $\omega_l$  on  $\omega_{atom}$  is usually performed with standard extremum seeking techniques since the resonance  $\Delta = 0$  is an absorption peak.
- ▶ We choose excited state with a long life-time (seconds). We propose a simple **feedback loop** for this problem.
- ▶ Atomic clocks can be used to make very precise portable magnetometers, and gravimeters.

# Bloch sphere



$$\Psi = \begin{pmatrix} \Psi_g \\ \Psi_e \end{pmatrix} \in \mathbb{C}^2$$

The probability conservation  $|\Psi_g|^2 + |\Psi_e|^2 = 1$  implies one can write

$$\Psi_g = \cos \beta > 0$$

$$\Psi_e = e^{i\phi} \sin \beta$$

$2\beta, \phi$  define Euler angles. Then consider the change of variables.

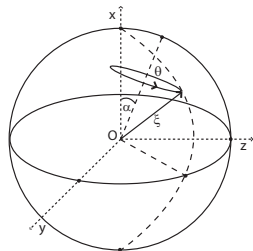
$$\Psi \mapsto \xi \in \mathbb{S}^2$$

Now we have:  $y = 2|\Psi_g|^2 - 1 = \xi_z$

# Bloch sphere

In the Bloch sphere the dynamics

$$i \frac{d}{dt} \Psi = \left( \frac{\Delta}{2} \sigma_z + \frac{\Omega_R}{2} \sigma_x \right) \Psi, \quad \Psi \in \mathbb{C}^2$$



is mirrored by

$$\frac{d}{dt} \xi = \left( \frac{\Omega_R}{2}, 0, \frac{\Delta}{2} \right) \wedge \xi, \quad \xi \in \mathbb{S}^2 \subset \mathbb{R}^3$$

with  $\Omega_R \gg 1$

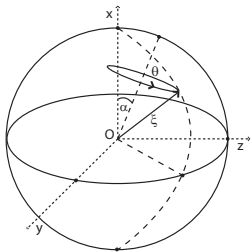
$\xi$  is **quickly** rotating around the vector

$(\cos \alpha, 0, \sin \alpha)$  (dashed line)

with angular velocity  $\frac{d}{dt} \theta = \sqrt{\Omega_R^2 + \Delta^2}$

where  $\left( \frac{\Omega_R}{2}, 0, \frac{\Delta}{2} \right) = \dot{\theta} (\cos \alpha, 0, \sin \alpha)$

# Interaction frame



$$\frac{d}{dt}\xi = \left(\frac{\Omega_R}{2}, 0, \frac{\Delta}{2}\right) \wedge \xi$$

$$\frac{\Delta}{\Omega_R} \ll 1$$

$$\frac{d}{dt}\theta = \sqrt{\Omega_R^2 + \Delta^2} \gg 1$$

Let us write the dynamics in the **interaction frame** (time-varying transformation):  $\xi \mapsto \eta$

$$\frac{d}{dt}\eta = \mathbf{0} + \left(\frac{d}{dt}\Delta\right) \frac{1}{2\Omega_R} (0, \cos\theta, -\sin\theta) \wedge \eta$$

$$\frac{d}{dt}\Delta = p\mathbf{u}$$

$$\mathbf{y} = \frac{\Delta}{\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z \quad (= \xi_z)$$

## Control law: principle of the proof

$$y = \frac{\Delta}{\Omega_R} \eta_x + \sin \theta \eta_y + \cos \theta \eta_z$$

Let  $K_f \ll \frac{d}{dt} \theta$  and

$$\frac{d}{dt} y_f = K_f (y - y_f) \Leftrightarrow "y_f \approx \frac{\Delta}{\Omega_R} \eta_x"$$

Set

$$u = Ky + K_2 y_f$$

$Ky$  is a resonant oscillating term that makes  $\eta \rightarrow (1, 0, 0)$

$K_2 y_f$  is a filtered term that makes  $\Delta \rightarrow 0$  once  $\eta$  has converged.

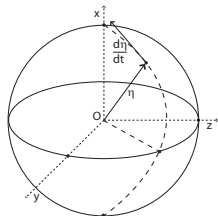
# Rotating wave (secular) approximation (RWA)

The integration of  $\cos(k\theta)$  and  $\sin(k\theta)$ ,  $k \in \mathbb{N}^*$  over the time  $t$  will produce terms of small amplitude rotating with high frequency and 0 mean.

We are going to neglect them and only keep the non-rotating terms (called “secular”). The standard rotating wave approximation consists in averaging the system over a period and **eliminate the terms rotating with frequency a multiple of  $\frac{d}{dt}\theta$  and with 0 mean.**

We keep constant terms and  $\sin^2(\theta)$ ,  $\cos^2(\theta)$  will be replaced by  $1/2$ .

# RWA and control law



First term:  $u = Ky$

$$\frac{d}{dt}\eta = \left[ Kp \left( \frac{\Delta}{2\Omega_R} \eta_x + \sin \theta \eta_y + \cos \theta \eta_z \right) \right] (0, \cos \theta, -\sin \theta) \wedge \eta$$

The averaged equation is

$$\frac{d}{dt}\eta = p \frac{K}{4\Omega_R} (\eta \wedge (1, 0, 0)) \wedge \eta$$

**Conclusion** If  $(K > 0)$  we have:  $\eta_x \rightarrow 1$

# RWA and control law

$$y = \frac{\Delta}{\Omega_R} \eta_x + \sin \theta \eta_y + \cos \theta \eta_z$$

## First term

And thus

$$\frac{d}{dt} \Delta = pu = pKy \rightarrow \frac{d}{dt} \Delta = pK \frac{\Delta}{\Omega_R}$$

$K > 0$  thus  $\Delta \rightarrow 0$  and the control law is not sufficient.

## Second term:

Let  $\frac{d}{dt} y_f = K_f (y - y_f)$

with  $\frac{d}{dt} \theta \approx \Omega_R \gg K_f$ . Thus  $y_f$  is an estimation of the filtered output

$$y_f \rightarrow \approx \frac{\Delta}{\Omega_R}$$

# RWA and control law

Take  $u = Ky + K_2 y_f$ . We have

$$\frac{d}{dt}\Delta = p(Ky + K_2 y_f)$$

The additional term  $K_2 y_f$  does not change the averaged  $\eta$  behaviour since

$$\begin{aligned} \frac{d}{dt}\eta &= [Kp\left(\frac{\Delta}{\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z\right) + K_2 y_f] \cdots \\ &\quad (0, \cos\theta, -\sin\theta) \wedge \eta \end{aligned}$$

so the averaged  $\eta_x \rightarrow 1$ .

# RWA and control law

But now we have for averaged  $\Delta$ :

$$\begin{aligned}\frac{d}{dt}\Delta &= p(K\Delta + K_2 y_f) \\ \frac{d}{dt}y_f &= K_f\left(\frac{\Delta}{\Omega_R} - y_f\right)\end{aligned}$$

$$\frac{d}{dt}\begin{pmatrix} \Delta \\ y_f \end{pmatrix} = \begin{pmatrix} pK & pK_2 \\ K_f/\Omega_R & -K_f/\Omega_R \end{pmatrix} \begin{pmatrix} \Delta \\ y_f \end{pmatrix}$$

Stable iff  $K_f > pK/\Omega_R$  (and recall  $K_f \ll \Omega_R$ ).

- ▶ The averaged system admits an **exponentially stable steady-state**.
- ▶  $\eta = (1, 0, 0)$ ,  $\Delta = 0$  is also a steady-state for the original system.

# Tuning of the gains via first-order approximation

Around  $\eta = (1, 0, 0) \Leftrightarrow \eta_x = 1$

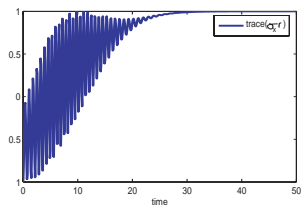
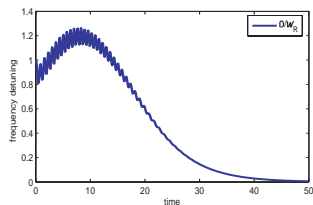
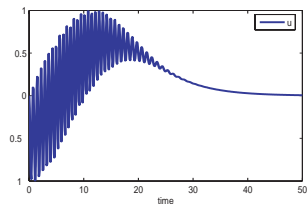
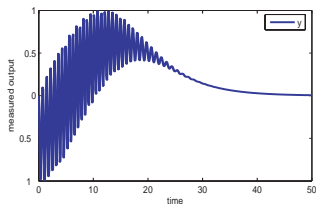
$$\begin{aligned}\frac{d}{dt}\eta_y &= -\frac{pK}{2\Omega_R}\eta_y, & \frac{d}{dt}\eta_z &= -\frac{pK}{2\Omega_R}\eta_z, \\ \frac{d}{dt}\Delta &= p\left(\frac{K}{2\Omega_R}\Delta + \frac{K_2}{2\Omega_R}y_f\right), & \frac{d}{dt}y_f &= K_f\left(\frac{\Delta}{2\Omega_R} - y_f\right)\end{aligned}$$

With  $\epsilon_1, \epsilon_2 \ll 1$ , a suitable choice of  $K, K_2, K_f$  gives the first-order approximation around the final state

$$\begin{aligned}\frac{d}{dt}\eta_y &= -\frac{\epsilon_1\Omega_R}{2}\eta_y, & \frac{d}{dt}\eta_z &= -\frac{\epsilon_1\Omega_R}{2}\eta_z, \\ \frac{d^2}{dt^2}\Delta &+ 2\Xi\epsilon_2\Omega_R\frac{d}{dt}\Delta + (\epsilon_2\Omega_R)^2\Delta &= 0.\end{aligned}$$

We take  $\epsilon_1 = 1/5, \epsilon_2 = 1/10$ .

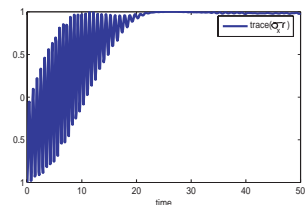
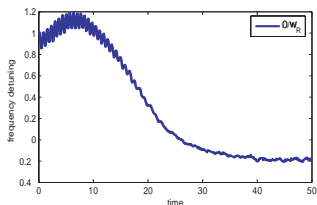
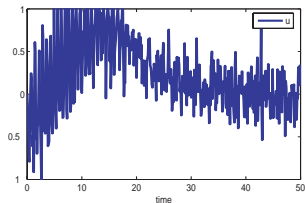
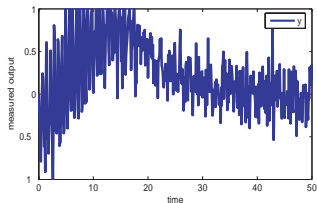
# Simulation with perfect measurements



Closed loop simulations of  $y$ ,  $u$ ,  $\Delta/(\Omega_R)$ ,  $\xi_X$

# Simulation with noisy and biased measurements:

$$\sigma = 2/10 \quad b = 2/10$$



Closed loop simulations of  $y$ ,  $u$ ,  $\Delta/(\Omega_R)$ ,  $\xi_x$

# Conclusion

- ▶ The feedback law is very simple and **can be achieved by a low-cost electronic circuit** which performs at high enough frequencies as the probe is usually a diode laser.
- ▶ Mirroring the dynamics on the Bloch sphere means the system is not entangled to its environment. In future work decoherence will be addressed.
- ▶ We hope the controller converges before the decoherence appears. Although it is designed for continuous operation clocks (like CPT), the clock could work in pulsed regime: the atoms are replaced by new ones in the next sequence.
- ▶ A way to avoid decoherence problem is to make the feedback on a single ion (stochastic problem) see (Mirrahimi, Rouchon [2008] on Arxiv).