A simple feedback-loop for a frequency lock on a narrow atomic transition

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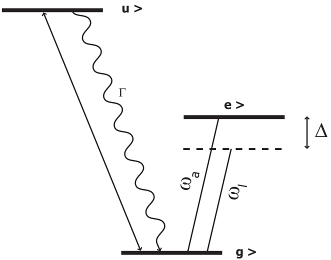
The control problem

Physical system:

Mono-atomic gaz. $E_e - E_g = \hbar\omega_{atom}$.

Mesurement:

- Photons emitted by the population of the ground state $|g\rangle$: $|u\rangle$ decays quickly to $|g\rangle$, and population $|u\rangle$ is negligible. Δ



Model

$$\imath \frac{d}{dt}\Psi = \left(\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x\right)\Psi, \quad \Psi = \begin{pmatrix}\Psi_g \\ \Psi_e\end{pmatrix} \in \mathbb{C}^2$$

- Ψ is the wave function: $|\Psi_g|^2 + |\Psi_e|^2 = 1$.
- $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices
- $\sigma_x^2 = 1$; $\sigma_x\sigma_y = \imath\sigma_z$..

Model

$$\imath \frac{d}{dt}\Psi = \left(\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x\right)\Psi, \quad \Psi = \begin{pmatrix}\Psi_g \\ \Psi_e\end{pmatrix} \in \mathbb{C}^2$$

Parameters

- Laser detuning $\Delta = \omega_{atom} - \omega_l$.
- Laser amplitude Ω_R/μ with $\mu > 0$.
- We have $1 << \Omega_R$ and $\Delta << \Omega_R$.

Control problem

$$\frac{d}{dt}\Delta = pu, \quad p > 0$$
$$y = 2|\Psi_g|^2 - 1$$

With control u, Ω_R find the atomic transition ω_{atom}:
We want $\Delta \to 0$.

Control law

$$\imath \frac{d}{dt}\Psi = \left(\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x\right)\Psi, \quad \Psi = \begin{pmatrix}\Psi_g \\ \Psi_e\end{pmatrix} \in \mathbb{C}^2$$

- set $\Omega_R = \bar{\Omega}_R$ to a positive constant
- $$u = Ky - \frac{K_2 K_f}{s + K_f}\, y$$
- $K, K_2, K_f > 0$ satisfy $\Omega_R \gg K_f > \frac{pK}{\Omega_R}$ and $\Omega_R^2 \gg pK, pK_2$

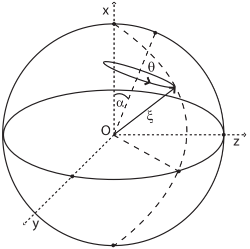
Result With such control we have locally

- $\Delta \to 0$.

Motivation: Atomic clocks

- Atoms in Dilute mono-atomic gases can be considered as perfect quantum systems with increasing energies $E_i = \hbar\omega_i$ depending only of the atomic species considered.
- Atomic clocks take advantage of the universality of $E_{eg} = \hbar\omega_{atom}$ to deliver a stable, periodic electromagnetic signal which frequency ω_{atom} in the GHz range.
- The laser frequency lock ω_l on ω_{atom} is usually performed with standard extremum seeking techniques since the resonance $\Delta = 0$ is an absorption peak.
- We choose excited state with a long life-time (seconds). We propose a simple feedback loop for this problem.
- Atomic clocks can be used to make very precise portable magnetometers, and gravitometers.

Bloch sphere



$$\Psi = \begin{pmatrix} \Psi_g \\ \Psi_e \end{pmatrix} \in \mathbb{C}^2$$

The probability conservation $|\Psi_g|^2 + |\Psi_e|^2 = 1$ implies one can write

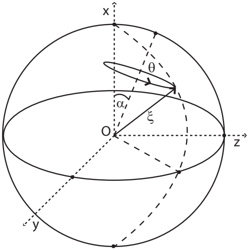
$$\Psi_g = \cos\beta > 0$$
$$\Psi_e = e^{\imath\phi}\sin\beta$$

$2\beta, \phi$ define Euler angles. Then consider the change of variables.

$$\Psi \mapsto \xi \in \mathbb{S}^2$$

Now we have: $y = 2|\Psi_g|^2 - 1 = \xi_z$

Bloch sphere



In the Bloch sphere the dynamics

$$\imath \frac{d}{dt}\Psi = \left(\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x\right)\Psi, \quad \Psi \in \mathbb{C}^2$$

is mirrored by

$$\frac{d}{dt}\xi = \left(\frac{\Omega_R}{2}, 0, \frac{\Delta}{2}\right) \wedge \xi, \quad \xi \in \mathbb{S}^2 \subset \mathbb{R}^3$$

with $\Omega_R >> 1$

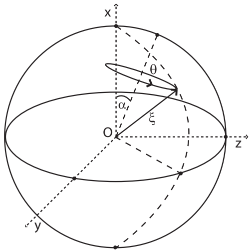
ξ is quickly rotating around the vector

$$(\cos\alpha, 0, \sin\alpha) \quad \text{(dashed line)}$$

with angular velocity $\frac{d}{dt}\theta = \sqrt{\Omega_R^2 + \Delta^2}$

where $\quad \left(\frac{\Omega_R}{2}, 0, \frac{\Delta}{2}\right) = \dot{\theta}(\cos\alpha, 0, \sin\alpha)$

Interaction frame



$$\frac{d}{dt}\xi = \left(\frac{\Omega_R}{2}, 0, \frac{\Delta}{2}\right) \wedge \xi$$

$$\frac{\Delta}{\Omega_R} << 1$$

$$\frac{d}{dt}\theta = \sqrt{\Omega_R^2 + \Delta^2} >> 1$$

Let us write the dynamics in the interaction frame (time-varying transformation): $\xi \mapsto \eta$

$$\begin{aligned} \frac{d}{dt}\eta &= 0 + \left(\frac{d}{dt}\Delta\right)\frac{1}{2\Omega_R}(0, \cos\theta, -\sin\theta) \wedge \eta \\ \frac{d}{dt}\Delta &= pu \\ y &= \frac{\Delta}{\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z \quad (= \xi_z) \end{aligned}$$

Control law: principle of the proof

$$y = \frac{\Delta}{\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z$$

Let $K_f \ll \frac{d}{dt}\theta$ and

$$\frac{d}{dt}y_f = K_f(y - y_f) \Leftrightarrow \text{``}y_f \approx \frac{\Delta}{\Omega_R}\eta_x\text{''}$$

Set

$$u = Ky + K_2 y_f$$

Ky is a resonant oscillating term that makes $\eta \to (1,0,0)$

$K_2 y_f$ is a filtered term that makes $\Delta \to 0$ once η has converged.

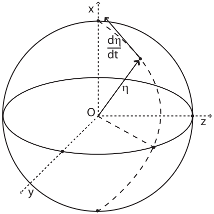
Rotating wave (secular) approximation (RWA)

The integration of $\cos(k\theta)$ and $\sin(k\theta), k \in \mathbb{N}^*$ over the time t will produce terms of small amplitude rotating with high frequency and 0 mean.

We are going to neglect them and only keep the non-rotating terms (called “secular"). The standard rotating wave approximation consists in averaging the system over a period and eliminate the terms rotating with frequency a multiple of $\frac{d}{dt}\theta$ and with 0 mean.

We keep constant terms and $\sin^2(\theta)$, $\cos^2(\theta)$ will be replaced by $1/2$.

RWA and control law



First term: $u = Ky$

$$\frac{d}{dt}\eta = [Kp(\frac{\Delta}{2\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z)](0, \cos\theta, -\sin\theta) \wedge \eta$$

The averaged equation is

$$\frac{d}{dt}\eta = p\frac{K}{4\Omega_R}(\eta \wedge (1, 0, 0)) \wedge \eta$$

Conclusion If $(K > 0)$ we have: $\eta_x \rightarrow 1$

RWA and control law

$$y = \frac{\Delta}{\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z$$

First term

And thus

$$\frac{d}{dt}\Delta = pu = pKy \rightarrow \frac{d}{dt}\Delta = pK\frac{\Delta}{\Omega_R}$$

$K > 0$ thus $\Delta \nrightarrow 0$ and the control law is not sufficient.

Second term:

Let $\frac{d}{dt}y_f = K_f(y - y_f)$

with $\frac{d}{dt}\theta \approx \Omega_R \gg K_f$. Thus y_f is an estimation of the filtered output

$$y_f \rightarrow \approx \frac{\Delta}{\Omega_R}$$

RWA and control law

Take $u = Ky + K_2 y_f$. We have

$$\frac{d}{dt}\Delta = p(Ky + K_2 y_f)$$

The additionnal term $K_2 y_f$ does not change the averaged η behaviour since

$$\frac{d}{dt}\eta = [Kp(\frac{\Delta}{\Omega_R}\eta_x + \sin\theta\eta_y + \cos\theta\eta_z) + K_2 y_f] \cdots (0, \cos\theta, -\sin\theta) \wedge \eta$$

so the averaged $\eta_x \to 1$.

RWA and control law

But now we have for averaged Δ:

$$\frac{d}{dt}\Delta = p(K\Delta + K_2 y_f)$$
$$\frac{d}{dt}y_f = K_f(\frac{\Delta}{\Omega_R} - y_f)$$

$$\frac{d}{dt}\begin{pmatrix}\Delta \\ y_f\end{pmatrix} = \begin{pmatrix} pK & pK_2 \\ K_f/\Omega_R & -K_f/\Omega_R \end{pmatrix}\begin{pmatrix}\Delta \\ y_f\end{pmatrix}$$

Stable iff $K_f > pK/\Omega_R$ (and recall $K_f << \Omega_R$).

- The averaged system admits an exponentially stable steady-state.
- $\eta = (1, 0, 0)$, $\Delta = 0$ is also a steady-state for the original system.

Tuning of the gains via first-order approximation

Around $\eta = (1,0,0) \Leftrightarrow \eta_x = 1$

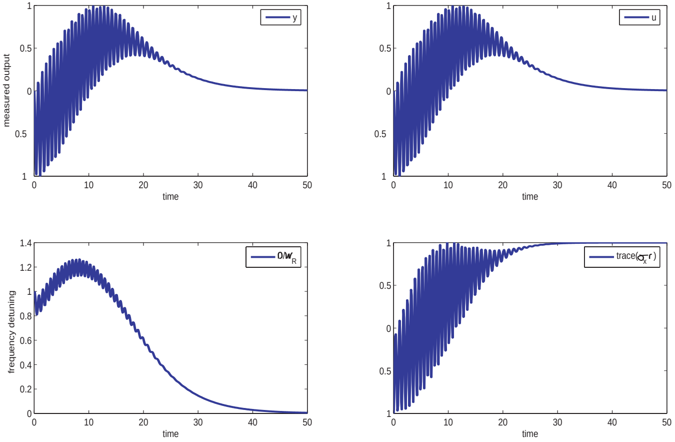
$$\frac{d}{dt}\eta_y = -\frac{pK}{2\Omega_R}\eta_y, \quad \frac{d}{dt}\eta_z = -\frac{pK}{2\Omega_R}\eta_z,$$
$$\frac{d}{dt}\Delta = p(\frac{K}{2\Omega_R}\Delta + \frac{K_2}{2\Omega_R}y_f), \quad \frac{d}{dt}y_f = K_f(\frac{\Delta}{2\Omega_R} - y_f)$$

With $\epsilon_1, \epsilon_2 \ll 1$, a suitable choice of K, K_2, K_f gives the first-order approximation around the final state

$$\frac{d}{dt}\eta_y = -\frac{\epsilon_1\Omega_R}{2}\eta_y, \quad \frac{d}{dt}\eta_z = -\frac{\epsilon_1\Omega_R}{2}\eta_z,$$
$$\frac{d^2}{dt^2}\Delta + 2\Xi\epsilon_2\Omega_R\frac{d}{dt}\Delta + (\epsilon_2\Omega_R)^2\Delta = 0.$$

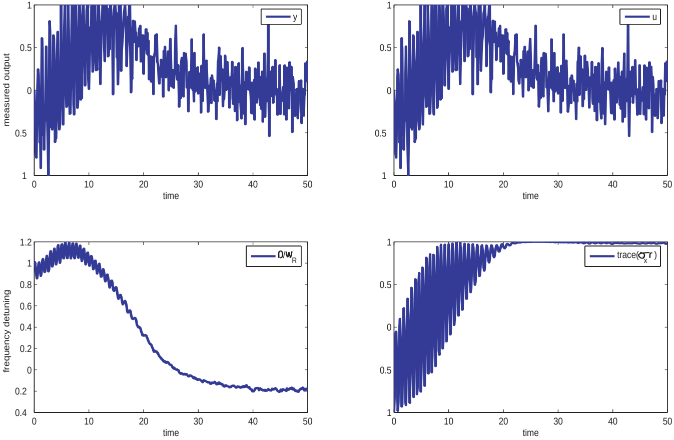
We take $\epsilon_1 = 1/5$, $\epsilon_2 = 1/10$.

Simulation with perfect measurements



Closed loop simulations of y, u, $\Delta/(\Omega_R)$, ξ_x

Simulation with noisy and biased measurements: $\sigma = 2/10$ $b = 2/10$



Closed loop simulations of y, u, $\Delta/(\Omega_R)$, ξ_x

Conlusion

- The feedback law is very simple and can be achieved by a low-cost electronic circuit which performs at high enough frequencies as the probe is usually a diode laser.
- Mirroring the dynamics on the Bloch sphere means the system is not entangled to its environment. In future work decoherence will be adressed.
- We hope the controller converges before the decoherence appears. Although it is desiegned for continuous operation clocks (like CPT), the clock could work in pulsed regime: the atoms are replaced by new ones in the next sequence.
- A way to avoid decoherence problem is to make the feedback on a single ion (stochastic problem) see (Mirrahimi, Rouchon [2008] on Arxiv).