A simple feedback-loop for a frequency lock on a narrow atomic transition ⋆

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Abstract: In this paper we consider an atomic system with a narrow transition probed with a user-controlled electromagnetic radiation, the basis of atomic clocks. We propose a simple feedback-loop in order to lock automatically the probe frequency on the atomic transition, allowing continuous clock operation without any preparation steps. The proof of the convergence of the feedback-loop is based on averaging arguments, with approximations compatible with realistic physical parameters. Numerical simulations illustrate the robustness of the proposed feedback-loop versus measurement noise and bias.

Keywords: Quantum systems, nonlinear stabilization, dynamic output-feedback, averaging, nonlinear oscillators, synchronization, atomic clock

1. INTRODUCTION

Dilute mono-atomic gases are very simple systems, in the sense their constituents (atoms), are perfectly identical and interact very weakly with each other. Atoms in such gases can be considered as perfect quantum systems, with a sequence of discrete energy states labeled $|i\rangle$, for $i \in \mathbb{N}$, with increasing energies $E_i = \hbar\omega_i$ depending only of the atomic species considered. Atomic clocks take advantage of the universality of these properties to deliver a stable, periodic electromagnetic signal which frequency ω is one the frequency differences $\omega_{ji} = \omega_j - \omega_i$. Compared to astronomical, mechanical and electromechanical clocks, atomic clocks have unprecedented long-term stability and suffer no ageing (see Audoin and Guinot [2001]).

The atomic gas interacts with electromagnetic radiation at a frequency ω only if there is a transition $i \rightarrow j$ with $\omega \approx \omega_{ji}$. Supposing that atoms are initially in the ground state $|i\rangle$, they can get to the excited state $|j\rangle$ by absorbing a photon at frequency ω. In a clock, absorption is measured and a feedback loop allows to lock the radiation frequency to the absorption peak. The width of the absorption peak $\delta\omega$ is limited by the natural lifetime in the upper state $|j\rangle$: $\delta\omega \times \tau_j \geq 1$. For a precise clock, one wants to minimize $\delta\omega$, and choose an excited state with a long lifetime (typically seconds). The maximum absorption rate is then limited by the time needed by the atom to spontaneously fall back to the initial state, leading to a poor continuous-operation absorption signal. To avoid that, atomic clocks work in a pulsed regime : atoms are initially prepared in state $|i\rangle$, then driven to the excited state $|j\rangle$ by one or two coherent pulses, and finally measured and replaced by new atoms for the next sequence.

A lot of effort is now being put into the development of cheaper, lighter and less power-consuming atomic clocks. Such devices have direct applications in portable geopositioning receivers, gravimeters, or magnetometers. Continuous-operation clocks, such as those based on coherence population trapping (CPT) resonance (see Vanier [2005]), are very desirable because they are simpler and require less equipment, power, and room. These clocks are based on the transmission peak (or dip) of a probe laser which frequency ω is close to the atomic system resonance. The laser frequency lock can be performed using standard extremum-seeking techniques, which rely on frequency modulation and synchronous measurement of the transmission (see Audoin and Guinot [2001]). In this paper, we propose a feedback scheme that allows to lock the frequency on a very narrow atomic transition, regardless of the initial state of the atoms. It is in principle suitable for continuous-operation clocks.

The paper is organized as follows. In section 2, we detail the system, its model and set the frequency lock problem as a stabilization problem via output feedback. In section 3, we present a simple dynamics output feedback scheme with gain design rules. Closed-loop simulations indicate a large convergence domain and also a good robustness versus measure noise an bias. In section 4, we propose, in a rather informal way, the basic step underlying a convergence proof and justify the gain design depicted in section 3.

2. THE SYSTEM

We consider a two-state quantum system with a ground state $|g\rangle$ and an excited state $|e\rangle$, defining a clock transition at a frequency $\omega_{eg} = \omega_{atom}$. ω_{atom} can be in the GHz range, (e.g.

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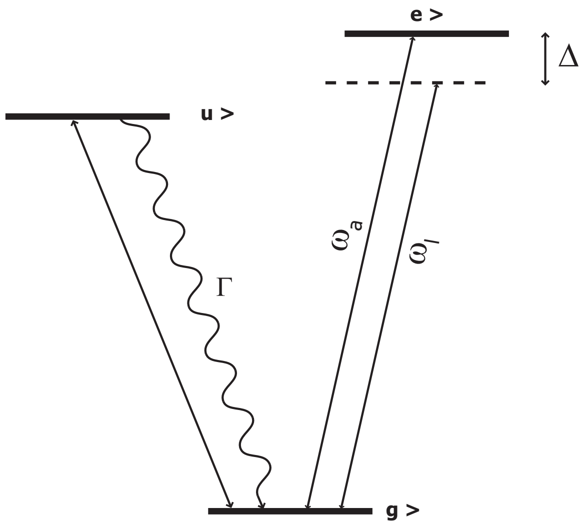

Fig. 1. Atomic states considered in the problem. Double-headed arrows indicate frequency differences. $g \to e$ is the clock transition, at a frequency ω_{eg}. $g \to u$ is the measurement transition, $|u\rangle$ decays rapidly to $|g\rangle$ with a rate Γ.

hyperfine transitions in caesium or rubidium, exploited by current frequency standards), up to the 10^{15} Hz range (single ion optical clocks, future optical lattice high performance clocks). The excited state is metastable, i.e. it has a very long lifetime. This transition is probed with a coherent radiation at frequency ω_l and amplitude A. It induces Rabi oscillations (photon absorption-emission cycle) at a frequency $\Omega_R = \mu A$, with μ a positive unknown constant. Typically Ω_R is in the kHz range is most clocks, but can be easily increased to MHz if necessary. As in quantum physics the measurement perturbs the system, the actual system consists here in a population of identical systems. The population of the ground state is continuously measured with a laser resonant with a second transition $g \to u$, where $|u\rangle$ is an unstable excited state which decays rapidly towards $|g\rangle$ with a decay rate Γ. Detection of fluorescence photons accompanying decays determines the population in state $|g\rangle$. Population in state $|u\rangle$ is negligible, so that the dynamics of the system is described by a two-level Schrödinger equation :

$$\imath \frac{d}{dt}\Psi = \left(\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x\right)\Psi, \qquad \Psi = \begin{pmatrix}\Psi_g \\ \Psi_e\end{pmatrix} \in \mathbb{C}^2 \qquad (1)$$

where we let

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -\imath\\ \imath & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

denote the Pauli matrices and where $\Delta = \omega_{atom} - \omega_l$ is the atom-probe pulsation detuning. We have the useful relations $\sigma_x^2 = 1$; $\sigma_x\sigma_y = \imath\sigma_z$ (with circular permutation). Through a weak measurement process (as already said earlier) we have access to the populations in real-time. This means that we can consider that $y =< \sigma_z\Psi, \Psi >= |\Psi_g|^2 - |\Psi_e|^2 = 2|\Psi_g|^2 - 1$ is the measured output of this system as the measurement is the ground state population $|\Psi_g|^2$ and the conservation of probability implies $|\Psi_g|^2 + |\Psi_e|^2 = 1$.

It is convenient to write the dynamics with the density matrix: let $\rho = \Psi\Psi^\dagger$ denote the complex matrix associated to the projector on the state Ψ. Supposing that the system is pure (meaning it is not entangled to its environment) implies both properties: $\mathrm{tr}(\rho) = \Psi_e^\dagger\Psi_e + \Psi_g^\dagger\Psi_g = 1$ and $\rho^2 = \Psi\Psi^\dagger\Psi\Psi^\dagger = \rho$. Thus rewriting (1) the system becomes

$$\frac{d}{dt}\rho = -\imath\left[\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x, \rho\right]$$
$$y = \mathrm{tr}(\sigma_z\rho)$$

where [,] is the commutator. The first controlled input is Ω_R, via the probe amplitude A. The second controlled input is associated to the probe frequency. Since we do not have access to ω_l precisely here, we cannot consider that Δ is the second input (Δ is not known here, only its time variation is). This means that the second controlled input is u with $\frac{d}{dt}\Delta = pu$ where p is a positive constant not known precisely (only its magnitude order is known) : the control input u acts on the time variation of Δ. The goal is to use the measured output y to steer, via A and u, the frequency detuning to 0. Of course we do not have access to the precise value of the parameters μ and p, to the detuning Δ and to the density matrix ρ. For a mathematical justification of this decoherence free model despite the presence of continuous measure, see, e.g., Mirrahimi and Rouchon [2006].

The automatic-control problem considered here is as follows: take the following system (two-state quantum system)

$$\begin{aligned}\frac{d}{dt}\rho &= -\imath\left[\frac{\Delta}{2}\sigma_z + \frac{\mu A}{2}\sigma_x, \rho\right]\\ \frac{d}{dt}\Delta &= pu\\ y &= \mathrm{tr}(\sigma_z\rho)\end{aligned} \qquad (2)$$

with controlled inputs A and u and with measured output y; design an output feedback law robust to more than 50% of relative-variation for μ and p such that Δ always converges to 0.

3. OUTPUT FEEDBACK AND SIMULATIONS

We propose the following feedback law: set $A = \overline{A}$ to a positive constant (since the probe amplitude is a controlled input) and take

$$u = (K_2 - K_1)y - \frac{K_2K_f}{s+K_f}\,y = \left(\frac{(K_2-K_1)s - K_1K_f}{s+K_f}\right)y \qquad (3)$$

where $s = \frac{d}{dt}$ is the Laplace variable, the gains K_1, K_2, K_f are strictly positive and satisfy $\Omega_R \gg K_f > \frac{p(K_2-K_1)}{\Omega_R}$ and $\Omega_R^2 \gg pK_1, pK_2$. The circuit realization of such simple transfert between y and u is elementary and thus can be implemented easily in practice, y varies with a characteristic frequency Ω_R. Notice that the zero of this transfert is unstable and thus is very different from the transfert of simple proportional/integral regulator.

For the closed-loop numerical simulation of figures 2 and 3 we take $A = 1$, $\mu = 2\pi$ and $p = 1$. The design of the 3 gains, K_1, K_2 and K_f, is given by (6) of section 4.4 with $\varepsilon_1 = 1/5$, $\varepsilon_2 = 1/10$ and $\Xi = 2$ and $\Omega_R \equiv \mu A$. The initial conditions are $\rho(0) = (I - \sigma_x)/2$ and $\Delta(0) = \Omega_R$ where I is the 2×2 identity matrix. These simulations illustrate the robustness of the proposed feedback loop versus measurement noise and constant bias.

4. CLOSED-LOOP CONVERGENCE ANALYSIS

A geometrical interpretation is given in the appendix.

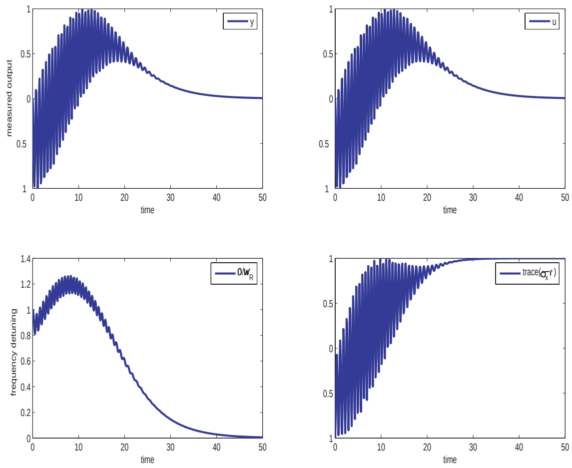

Fig. 2. Closed-loop simulation of (2) with output feedback (3); perfect measurement. The 4 graphics represent y, u, Δ/Ω_R and $\mathrm{tr}(\sigma_x\rho)$.

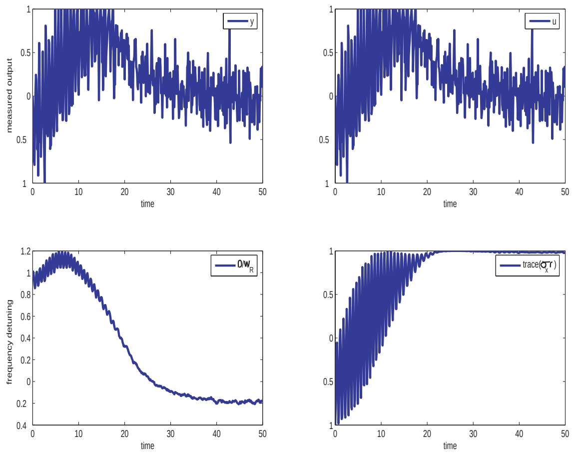

Fig. 3. Closed-loop simulation of (2) with output feedback (3); measurement with an additive gaussian with noise of RMS $1/5$ and a constant bias of $1/5$. The 4 graphics represent y, u, Δ/Ω_R and $\mathrm{tr}(\sigma_x\rho)$. Notice that the measurement bias induces a bias for Δ/Ω_R.

4.1 Change of variables

In order to have simpler formulas we consider some changes of variables. Define the angles α and θ by the relations

$$e^{\imath\alpha} = \frac{\Omega_R + \imath\Delta}{\sqrt{\Omega_R^2+\Delta^2}}, \qquad \frac{d}{dt}\theta = \sqrt{\Omega_R^2+\Delta^2}$$

Consider the following change of coordinates (new frame)

$$\rho = e^{\imath\frac{\alpha\sigma_y}{2}} e^{-\imath\frac{\theta\sigma_x}{2}} \zeta e^{\imath\frac{\theta\sigma_x}{2}} e^{-\imath\frac{\alpha\sigma_y}{2}}.$$

First, in order to express the system in the new variables, we introduce some useful formulas : for any real a we have (for instance with σ_x)

$$e^{\imath a\sigma_x} = \cos a + \imath \sin a\ \sigma_x \quad \text{and thus} \quad e^{\imath a\sigma_x}\sigma_y = \sigma_y e^{-\imath a\sigma_x}$$

which can be completed performing circular permutations on $\sigma_x, \sigma_y, \sigma_z$. This proves that the output writes

$$\begin{aligned}
y &= \mathrm{tr}\left(e^{\imath\frac{\theta\sigma_y}{2}} e^{-\imath\frac{\alpha\sigma_x}{2}} \sigma_z e^{\imath\frac{\alpha\sigma_x}{2}} e^{-\imath\frac{\theta\sigma_y}{2}} \zeta\right)\\
&= \mathrm{tr}\big((\cos\frac{\theta}{2} + \imath\sin\frac{\theta}{2}\sigma_x)(\cos\alpha - \imath\sin\alpha\sigma_y)..\\
&\quad ..\sigma_z(\cos\frac{\theta}{2} - \imath\sin\frac{\theta}{2}\sigma_x)\zeta\big)\\
&= \mathrm{tr}\big((\sin\alpha(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2})\sigma_x + 2\cos\alpha\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sigma_y\\
&\qquad + \cos\alpha(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}))\zeta\big)\\
&= \mathrm{tr}\left((\sin\alpha\sigma_x + \cos\alpha(\sin\theta\sigma_y + \cos\theta\sigma_z))\zeta\right)
\end{aligned}$$

Another useful feature is that for any real a and 2×2 matrix M

$$\begin{aligned}
e^{\imath\frac{a}{2}\sigma_x}[\sigma_y, M]e^{-\imath\frac{a}{2}\sigma_x} &= [e^{\imath a\sigma_x}\sigma_y, e^{\imath\frac{a}{2}\sigma_x}Me^{-\imath\frac{a}{2}\sigma_x}]\\
e^{\imath\frac{a}{2}\sigma_x}[\sigma_x, M]e^{-\imath\frac{a}{2}\sigma_x} &= [\sigma_x, e^{\imath\frac{a}{2}\sigma_x}Me^{-\imath\frac{a}{2}\sigma_x}]
\end{aligned} \tag{4}$$

which can be completed performing circular permutations on $\sigma_x, \sigma_y, \sigma_z$. Using (2) we have:

$$\begin{aligned}
\frac{d}{dt}\zeta &= \frac{d}{dt}(e^{\imath\frac{\theta\sigma_x}{2}} e^{-\imath\frac{\alpha\sigma_y}{2}} \rho e^{\imath\frac{\alpha\sigma_y}{2}} e^{\imath\frac{-\theta\sigma_x}{2}})\\
&= \imath\frac{\dot\theta}{2}[\sigma_x, \zeta] + e^{\imath\frac{\theta\sigma_x}{2}}\big(-\imath\frac{\dot\alpha}{2}[\sigma_y, e^{-\imath\frac{\alpha\sigma_y}{2}}\rho e^{\imath\frac{\alpha\sigma_y}{2}}]\\
&\qquad - e^{-\imath\frac{\alpha\sigma_y}{2}}[\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x, \rho]e^{\imath\frac{\alpha\sigma_y}{2}}\big)e^{-\imath\frac{\theta\sigma_x}{2}}
\end{aligned}$$

Using the formula (4) we have on the one hand

$$-\imath\frac{\dot\alpha}{2}e^{\imath\frac{\theta\sigma_x}{2}}[\sigma_y, e^{-\imath\frac{\alpha\sigma_y}{2}}\rho e^{\imath\frac{\alpha\sigma_y}{2}}]e^{\imath\frac{\theta\sigma_x}{2}} = -\imath\frac{\dot\alpha}{2}[e^{\imath\theta\sigma_x}\sigma_y, \zeta]$$

and still using (4) we have on the other hand

$$\begin{aligned}
&e^{-\imath\frac{\alpha\sigma_y}{2}}[\frac{\Delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x, \rho]e^{\imath\frac{\alpha\sigma_y}{2}}\\
&= [\frac{\Delta}{2}(\cos\alpha\ \sigma_z + \sin\alpha\ \sigma_x)\\
&\quad + \frac{\Omega_R}{2}(\cos\alpha\ \sigma_x - \sin\alpha\ \sigma_z), e^{-\imath\frac{\alpha\sigma_y}{2}}\rho e^{\imath\frac{\alpha\sigma_y}{2}}]
\end{aligned}$$

But by definition of α we have

$$\cos\alpha = \frac{\Omega_R}{\sqrt{\Omega_R^2+\Delta^2}} \quad \text{and} \quad \sin\alpha = \frac{\Delta}{\sqrt{\Omega_R^2+\Delta^2}}$$

Using also $\dot\theta = \sqrt{\Omega_R^2+\Delta^2}$. In the new variables the system (2) boils down to

$$\begin{aligned}
\frac{d}{dt}\zeta &= -\imath\frac{\dot\alpha}{2}[\cos\theta\sigma_y - \sin\theta\sigma_z, \zeta]\\
\frac{d}{dt}\Delta &= pu\\
y &= \mathrm{tr}\left((\sin\alpha\sigma_x + \cos\alpha(\sin\theta\sigma_y + \cos\theta\sigma_z))\zeta\right)
\end{aligned}$$

where

$$\frac{d}{dt}\alpha = \cos\alpha\frac{d}{dt}\Delta/\sqrt{\Delta^2+\Omega_R^2}$$

since Ω_R (i.e. A) is kept constant. This last equality is obtained writing $\frac{d}{dt}\alpha = -\imath(\frac{d}{dt}e^{\imath\alpha})e^{-\imath\alpha}$.

4.2 Closed-loop dynamics in the new variables

With the feedback (3), the closed-loop system reads

$$\begin{aligned}
\frac{d}{dt}\zeta &= -\imath\left(\frac{p\Omega_R(-K_1y + K_2(y-y_f))}{2(\Delta^2+\Omega_R^2)}\right)[\cos\theta\sigma_y - \sin\theta\sigma_z, \zeta]\\
\frac{d}{dt}\Delta &= p(-K_1y + K_2(y-y_f))\\
\frac{d}{dt}y_f &= K_f(y-y_f)\\
y &= \mathrm{tr}\left((\sin\alpha\sigma_x + \cos\alpha(\sin\theta\sigma_y + \cos\theta\sigma_z))\zeta\right).
\end{aligned}$$

Assume that $\Delta \ll \Omega_R$ (which means the system is close to resonance). We have up to second order terms in Δ/Ω_R,

$$\begin{aligned}
\frac{d}{dt}\zeta &= -\imath\left(\frac{p(-K_1 y + K_2(y-y_f))}{2\Omega_R}\right)[\cos\theta\sigma_y - \sin\theta\sigma_z, \zeta] \\
\frac{d}{dt}\Delta &= p(-K_1 y + K_2(y-y_f)) \\
\frac{d}{dt}y_f &= K_f(y-y_f) \\
y &= \operatorname{tr}\left((\frac{\Delta}{\Omega_R}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right)
\end{aligned}$$

where $\frac{d}{dt}\theta = \Omega_R \gg K_f, \frac{K_1}{\Omega_R}, \frac{K_2}{\Omega_R}$. With

$$\bar{K}_1 = \frac{K_1}{\Omega_R}, \quad \bar{K}_2 = \frac{K_2}{\Omega_R}, \quad \bar{\Delta} = \frac{\Delta}{\Omega_R}$$

we have the following system

$$\begin{aligned}
\frac{d}{dt}\zeta &= -\imath\frac{p((\bar{K}_2-\bar{K}_1)\operatorname{tr}\left((\bar{\Delta}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right) + \bar{K}_2 y_f)}{2} \\
&\quad \ldots[\cos\theta\sigma_y - \sin\theta\sigma_z, \zeta] \\
\frac{d}{dt}\bar{\Delta} &= p((\bar{K}_2-\bar{K}_1)\operatorname{tr}\left((\bar{\Delta}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right) + \bar{K}_2 y_f \\
\frac{d}{dt}y_f &= K_f(\operatorname{tr}\left((\bar{\Delta}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right) - y_f) \\
y &= \operatorname{tr}\left((\bar{\Delta}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right)
\end{aligned}$$

4.3 Rotating wave approximation

We have $\frac{d}{dt}\theta \gg K_f, \bar{K}_1, \bar{K}_2$. Thus $\frac{d}{dt}\theta$ is a high frequency. The integration of $\exp(\imath k\theta), k \in \mathbb{N}^*$ over the time t will produce terms of small amplitude rotating with high frequency and 0 mean. We are going to neglect them and only keep the non-rotating terms (called "secular"). Indeed the standard rotating wave approximation consists in averaging the system over a period and eliminate the terms rotating with frequency a multiple of $\frac{d}{dt}\theta$ and with 0 mean (see, e.g., Haroche and Raimond [2006] for a physicist point of view or Arnold [1976], Guckenheimer and Holmes [1983] for a more formal exposure). The above system reads (after averaging):

$$\begin{aligned}
\frac{d}{dt}\zeta &= -\imath\frac{p(\bar{K}_2-\bar{K}_1)}{4}\left(\operatorname{tr}(\sigma_z\zeta)[\sigma_y,\zeta] - \operatorname{tr}(\sigma_y\zeta)[\sigma_z,\zeta]\right) \\
\frac{d}{dt}\bar{\Delta} &= p((\bar{K}_2-\bar{K}_1)\bar{\Delta}\operatorname{tr}(\sigma_x\zeta) + \bar{K}_2 y_f) \\
\frac{d}{dt}y_f &= K_f(\bar{\Delta}\operatorname{tr}(\sigma_x\zeta) - y_f) \\
y &= \bar{\Delta}\operatorname{tr}(\sigma_x\zeta)
\end{aligned} \tag{5}$$

where we used the fact that $\sin\theta$, $\cos\theta$, and $(\sin\theta\cos\theta)$ are equal to 0 over a period and $\sin^2\theta$ and $\cos^2\theta$ are equal to $\frac{1}{2}$. This system has a triangular structure where the evolution of ζ is decoupled from the evolution of $\bar{\Delta}$ and y_f. This is due to the fact that only y is interfering with the rotating term $[\cos\theta\sigma_y - \sin\theta\sigma_z, \zeta]$ (and not y_f). Since

$$\frac{d}{dt}\operatorname{tr}(\sigma_x\zeta) = \frac{p(\bar{K}_2-\bar{K}_1)}{2}(\operatorname{tr}(\sigma_y\zeta)^2 + \operatorname{tr}(\sigma_z\zeta)^2) \geq 0$$

necessarily, $\operatorname{tr}(\sigma_x\zeta)$ converges to 1 and thus ζ to $(I+\sigma_x)/2$. [1] Physically, it means that all the atoms of the gaz converge to the same state. Once ζ has converged to $(I+\sigma_x)/2$, $\bar{\Delta}$ and y_f obey

$$\frac{d}{dt}\bar{\Delta} = p((\bar{K}_2-\bar{K}_1)\bar{\Delta} + \bar{K}_2 y_f), \quad \frac{d}{dt}y_f = K_f(\bar{\Delta} - y_f)$$

[1] See the appendix for a geometrical interpretation.

thus y_f is an estimation of $\bar{\Delta}$. This system is stable as soon as the Jacobian matrix $\begin{pmatrix} p(\bar{K}_2-\bar{K}_1) & p\bar{K}_2 \\ K_f & -K_f \end{pmatrix}$ has its eigenvalues with negative real part, i.e. $K_f > p(\bar{K}_2-\bar{K}_1)$. In this case, the averaged system admits an exponentially stable steady-state that is also an equilibrium of the original system, thus the steady-state is also exponentially stable for the original system (cf. Khalil [1992], Theorem 8.3).

We proved (using averaging arguments) that with the feedback control law (3) and a suitable choice of the gains K_f, K_1, K_2, the system (2) is such that the laser de-tuning Δ converges to 0 indeed. By the way we also proved that the variable $e^{\imath\frac{\theta\sigma_x}{2}}e^{-\imath\frac{\alpha\sigma_y}{2}}\rho e^{\imath\frac{\alpha\sigma_y}{2}}e^{-\imath\frac{\theta\sigma_x}{2}}$ converges to $(I+\sigma_x)/2$.

4.4 First-order approximation and tuning of the gains

The convergence analysis is local and based on averaging arguments. It suggests the following design for the gain K_1, K_2, K_f based on the tangent linear system around $\zeta = (I+\sigma_x)/2$. Let $Y = \operatorname{tr}(\sigma_y\zeta)$ and $Z = \operatorname{tr}(\sigma_z\zeta)$. (They are the coordinates of ζ on the Bloch sphere, see the appendix). The first-order approximation of the system around $\zeta = (I+\sigma_x)/2$ writes:

$$\begin{aligned}
&\frac{d}{dt}Y = -\frac{p(\bar{K}_2-\bar{K}_1)}{2}Y, \quad \frac{d}{dt}Z = -\frac{p(\bar{K}_2-\bar{K}_1)}{2}Z, \\
&\frac{d}{dt}\bar{\Delta} = p((\bar{K}_2-\bar{K}_1)\bar{\Delta} + \bar{K}_2 y_f), \quad \frac{d}{dt}y_f = K_f(\bar{\Delta} - y_f)
\end{aligned}$$

Take ε_1 and ε_2 two small parameters $0 < \varepsilon_1, \varepsilon_2 \ll 1$, and $\Xi > 0$ and set

$$K_1 = \frac{(\varepsilon_2\Omega_R)^2}{p(\varepsilon_1 + 2\Xi\varepsilon_2)}, \quad K_2 = K_1 + \frac{\varepsilon_1\Omega_R^2}{p}, \quad K_f = (\varepsilon_1 + 2\Xi\varepsilon_2)\Omega_R. \tag{6}$$

Then the tangent linear system around the final state $(I+\sigma_x)/2$ reads

$$\begin{aligned}
&\frac{d}{dt}Y = -\frac{\varepsilon_1\Omega_R}{2}Y, \quad \frac{d}{dt}Z = -\frac{\varepsilon_1\Omega_R}{2}Z, \\
&\frac{d^2}{dt^2}\bar{\Delta} + 2\Xi\varepsilon_2\Omega_R\frac{d}{dt}\bar{\Delta} + (\varepsilon_2\Omega_R)^2\bar{\Delta} = 0.
\end{aligned}$$

It implies that the system converges to $(I+\sigma_x)/2$ indeed, since Y, Z tend to 0, and $X^2+Y^2+Z^2 = 1$ (see the appendix). The probe detuning Δ tends to 0. Eventually all the atoms are in the same state and the probe is in resonance with the system.

5. CONCLUSION

We proposed a controller which allows to continuously tune a probe radiation to the transition frequency of a two-state quantum system. The feedback law is very simple and can be performed by a low-cost electronic circuit.

In order to obtain convergence (as it is done in this paper), it is necessary that the initial detuning $\Delta(0)$ is smaller than Ω_R, otherwise the transition is not probed. In the case of optical clocks, this can be difficult, so we can increase (temporary) Ω_R to the MHz range, making it easier to find the transition . Ω_R is also the characteristic frequency for the modulation of the probe frequency and the response of the electronic circuit to ensure the key resonance conditions highlighted by the closed-loop convergence analysis. For optical clocks, the probe is typically a diode laser, which current can easily be modulated at such frequencies. Electronic circuits routinely work at these frequencies.

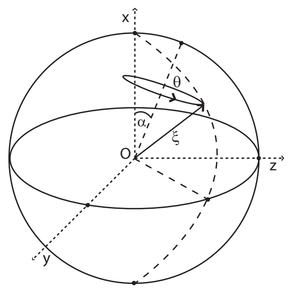

Fig. 4. The density matrix ρ is "mirrored" by ξ on the Bloch sphere. It is rotating around the axis $(\cos\alpha, 0, \sin\alpha)$ (dashed line) where $(\frac{\Omega_R}{2}, 0, \frac{\Delta}{2}) = \dot{\theta}(\cos\alpha, 0, \sin\alpha)$ with angular velocity $\dot{\theta}$.

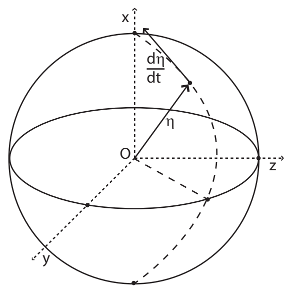

Fig. 5. The density matrix in the new frame ζ is "mirrored" by η on the Bloch sphere. The averaged dynamics of η is a gradient dynamics such that X converges to 1. Indeed $\frac{d}{dt}\eta$ is always pointing "north".

In this paper we supposed that the two-state system was pure (i.e, it is not entangled to its environment), and we did not take into account the decoherence due to the environment. In this case the system does not stay on the boundary of the Bloch sphere. This issue will be addressed in future work and can be treated adding Lindblad terms in the differential equations of ρ.

6. APPENDIX

6.1 Geometrical interpretation : the Bloch sphere

The Bloch sphere is a geometrical representation of the state space of a (pure) two-level quantum mechanical system. An important property is that any density matrix ρ of such a system is a projector with unit trace and can be written

$$\rho = \frac{1 + X\sigma_x + Y\sigma_y + Z\sigma_z}{2}, \text{ with } \xi = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{S}^2$$

where 1 denotes the identity 2×2 matrix. Thus we have the useful formulas:

$$\operatorname{tr}(\sigma_x\rho) = X, \quad \operatorname{tr}(\sigma_y\rho) = Y, \quad \operatorname{tr}(\sigma_z\rho) = Z$$

The output of the system is the Z-coordinate on the Bloch sphere. The commutation operation $-\imath[\sigma_x, \rho]$ corresponds to the wedge product $e_x \wedge \xi$ (circular permutations allow to complete the correspondences).

6.2 Interpretation of the convergence analysis

The dynamics of ξ is

$$\frac{d}{dt}\xi = (\frac{\Omega_R}{2}, 0, \frac{\Delta}{2}) \wedge \xi$$

This is a rotation with angular velocity $\frac{d}{dt}\theta$ around an axis lying in the (x,z)-plane (see fig 4). The angle between the rotation axis and the x-axis is α. This explains the change of variables of section 4.1 since we took $e^{\imath\alpha} = \frac{\Omega_R + \imath\Delta}{\sqrt{\Omega_R^2 + \Delta^2}}$, $\quad \frac{d}{dt}\theta = \sqrt{\Omega_R^2 + \Delta^2}$. After having made this change of variables the dynamics is written in a new frame linked to the rotation axis. ζ is the density matrix in this new frame. The dynamics in the new frame is easier to compute with density matrix than directly in the Bloch sphere. Our main hypothesis is that $\Delta \ll \Omega_R$. Up to second order terms in Δ/Ω_R, we have $\alpha = \Delta/\Omega_R$, $\sin\alpha = \alpha$ and $\cos\alpha = 1$. As proved previously, the system writes

$$\begin{aligned} \frac{d}{dt}\zeta &= -\imath\left(\frac{pu}{2\Omega_R}\right)[\cos\theta\sigma_y - \sin\theta\sigma_z, \zeta] \\ \frac{d}{dt}\Delta &= pu \\ \frac{d}{dt}y_f &= K_f(\operatorname{tr}\left((\frac{\Delta}{\Omega_R}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right) - y_f) \\ y &= \operatorname{tr}\left((\frac{\Delta}{\Omega_R}\sigma_x + \sin\theta\sigma_y + \cos\theta\sigma_z)\zeta\right) \end{aligned}$$

y_f filters the high frequencies (since $K_f \ll \Omega_R$) and provides an estimation of $\frac{\Delta}{\Omega_R}\operatorname{tr}(\sigma_x\zeta)$. We took $u = (K_2 - K_1)y - K_2 y_f$. Thus u is made of a high frequency rotating term $(K_2 - K_1)y$ and of a slowly-varying term $K_2 y_f$ which does not provide any secular term. Indeed in the secular approximation of $\frac{d}{dt}\zeta$ (closed-loop) only the oscillating part $(K_2 - K_1)y$ plays a role since the averaged equation writes (see eq (5))

$$\frac{d}{dt}\zeta = -\imath\frac{p(\bar{K}_2 - \bar{K}_1)}{4}(\operatorname{tr}(\sigma_z\zeta)[\sigma_y, \zeta] - \operatorname{tr}(\sigma_y\zeta)[\sigma_z, \zeta])$$

We are going to prove it is a gradient dynamics. Indeed let us write it in the Bloch sphere :

$$\text{Set } \zeta = \frac{1 + X\sigma_x + Y\sigma_y + Z\sigma_z}{2}, \text{ and } \eta = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{S}^2$$

η has the following dynamics :

$$\frac{d}{dt}\eta = p\frac{\bar{K}_2 - \bar{K}_1}{2}(\eta \wedge e_x) \wedge \eta$$

$\frac{d}{dt}\eta$ is a vector pointing towards e_x on the sphere (see fig 5). It is equal to zero only when $\eta = e_x$. It implies that $\frac{d}{dt}\operatorname{tr}(\sigma_x\rho) = \frac{d}{dt}X$ is positive and X converges to 1. Since η belongs to a unit sphere, it means that Y and Z converge to 0 and thus ζ converges to $(I + \sigma_x)/2$.

This is true only if $K_2 - K_1 \geq 0$. But the averaged dynamics (eq (5)) for Δ writes $\frac{d}{dt}\Delta = p((K_2 - K_1)\Delta\operatorname{tr}(\sigma_x\zeta) + K_2 y_f)$. And thus the filtered output y_f must be part of the feedback law otherwise Δ can not converge to zero.

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